STEP, 2012, Paper 2, Q9 - Solution (15/7/18; 2 pages)

[This question was very straightforward for STEP; especially STEP 2. The ER mentions that it was the most highly-scoring question in the paper, although only 40% of candidates tackled it (this still counted as 'popular' though). Given the limited theory involved, and the fact that everything is 'show that', it is a very attractive question.]

Let T be the time when the ball is level with the net.

Then $2h - usin\alpha.T - \frac{g}{2}T^2 > h$ (vertical distance)

and $ucos\alpha.T = a$ (horiz. distance)

Hence $2h - atan\alpha - \frac{g}{2}\left(\frac{a}{ucos\alpha}\right)^2 > h$

$\Rightarrow h - atan\alpha > \frac{g}{2}.\frac{a^2}{u^2cos^2\alpha}$

$\Rightarrow \frac{2(h-atan\alpha)}{ga^2sec^2\alpha} > \frac{1}{u^2}$, as required (A)

Let T_L be the time when the ball lands.

Then (vert.) $2h - usin\alpha.T_L - \frac{g}{2}.{T_L}^2 = 0$ (1)

and (horiz.) $ucos\alpha.T_L < b$ (2)

From (1), ${T_L}^2 + \frac{2usin\alpha T_L}{g} - \frac{4h}{g} = 0$

and hence $T_L = \frac{-\frac{2usin\alpha}{g} + \sqrt{\frac{4u^2sin^2\alpha}{g^2} + \frac{16h}{g}}}{2}$ (as $T_L > 0$)

Then, from (2), $\frac{b}{ucos\alpha} > \frac{-usin\alpha}{g} + \frac{1}{g}\sqrt{u^2sin^2\alpha + 4hg}$

$\Rightarrow \frac{bg}{ucos\alpha} + usin\alpha > \sqrt{u^2sin^2\alpha + 4gh}$, as required (B)

[initial aim is to eliminate u]

From (B), $u^2sin^2\alpha + 4gh < \frac{b^2g^2}{u^2cos^2\alpha} + u^2sin^2\alpha + 2bgtan\alpha$

$\Rightarrow 4gh - 2bgtan\alpha < \frac{b^2g^2}{u^2cos^2\alpha}$

$\Rightarrow \frac{1}{u^2cos^2\alpha} > \frac{4h-2btan\alpha}{b^2g}$ (3)

From (A), $\frac{1}{u^2cos^2\alpha} < \frac{2(h-atan\alpha)}{ga^2}$ (4)

(3)&(4) then $\Rightarrow \frac{2(h-atan\alpha)}{ga^2} > \frac{4h-2btan\alpha}{b^2g}$

$\Rightarrow b^2(h - atan\alpha) > (2h - btan\alpha)a^2$

[it is probably not necessary to point out (here and elsewhere) that we are multiplying by obviously positive quantities, so that the direction of the inequality is not affected]

$\Rightarrow b^2h - 2ha^2 > tan\alpha(ab^2 - ba^2)$

$\Rightarrow tan\alpha < \frac{h(b^2-2a^2)}{ab(b-a)}$, since $b > a$ [this does have to be pointed out], as required