STEP 2012, Paper 2, Q13 - Solutions (15/7/18; 2 pages)

Let X be the number of supermarkets in a circular region of radius y, so that $X \sim Po(k\pi y^2)$

$P(X = 0) = e^{-k\pi y^2}$

Consider a circle of radius y about the chosen point.

$P(Y < y) = 1 - P(Y > y) = 1 - P(X = 0) = 1 - e^{-k\pi y^2}$

[This is the cumulative distribution function of Y]

pdf of Y $= \frac{d}{dy}\left(1 - e^{-k\pi y^2}\right) = 2\pi y k e^{-\pi k y^2}$, as required

$E(Y) = \int_0^\infty y(\,2\pi y k e^{-\pi k y^2})dy$

Integrating by Parts, $\int\ 2\pi y k e^{-\pi k y^2} dy = -e^{-\pi k y^2}$,

so that $E(Y) = \left[y\left(-e^{-\pi k y^2}\right)\right]_0^\infty - \int_0^\infty\ - e^{-\pi k y^2} dy$

$= (0 - 0) + \int_0^\infty\ e^{-\pi k y^2} dy$

We are given that $\int_0^\infty\ e^{-\frac{x^2}{2}} dx = \sqrt{\frac{\pi}{2}}$

Let $x = y\sqrt{2\pi k}$

Then $dx = dy\sqrt{2\pi k}$ and $\int_0^\infty\ e^{-\frac{x^2}{2}} dx = \sqrt{2\pi k}\int_0^\infty\ e^{-\pi k y^2} dy$,

so that $E(Y) = \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi k}} = \frac{1}{2\sqrt{k}}$

$Var(X) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \int_0^\infty y^2(\,2\pi yke^{-\pi ky^2})dy$

By Parts again, $E(Y^2) = \left[y^2\left(-e^{-\pi ky^2}\right)\right]_0^\infty - \int_0^\infty \; -2ye^{-\pi ky^2}dy$

$= (0-0) + \frac{1}{\pi k}\int_0^\infty \; 2\pi yke^{-\pi ky^2}dy = \frac{1}{\pi k}$, as $2\pi yke^{-\pi ky^2}$ is the pdf

So $Var(Y) = \frac{1}{\pi k} - \left(\frac{1}{2\sqrt{k}}\right)^2 = \frac{4-\pi}{4\pi k}$, as required