STEP 2011, Paper 3, Q8 – Solution (4 pages; 12/6/18)

$u + iv = \frac{1+i(x+iy)}{i+x+iy} = \frac{(1-y)+ix}{x+i(1+y)} = \frac{[(1-y)+ix][x-i(1+y)]}{x^2+(1+y)^2}$

$= \frac{(1-y)x+x(1+y)+i(x^2-(1-y^2))}{x^2+(1+y)^2}$

Then, equating real & imaginary parts,

$u = \frac{2x}{x^2+(1+y)^2}$ & $v = \frac{x^2+y^2-1}{x^2+(1+y)^2}$

[This method guarantees that the real and imaginary parts of w can be written down fairly quickly. If instead we write $w(i+z) = 1 + iz$, and then equate real and imaginary parts, we end up with some awkward simultaneous equations.]

(i) [We can easily show that, with $y = 0, u^2 + v^2 = 1$. However, we need to establish precisely what part of the circle is actually used.]

Let $x = \tan\left(\frac{\theta}{2}\right)$, so that $u = \frac{2\tan\left(\frac{\theta}{2}\right)}{sec^2\left(\frac{\theta}{2}\right)} = 2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)$

$= sin\theta$

And $v = \frac{\tan^2\left(\frac{\theta}{2}\right)-1}{sec^2\left(\frac{\theta}{2}\right)} = \sin^2\left(\frac{\theta}{2}\right) - cos^2\left(\frac{\theta}{2}\right) = -cos\theta$

Thus $u^2 + v^2 = 1$

With $-\infty < x < \infty, -\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}$ or $-\frac{\pi}{2} + \pi < \frac{\theta}{2} < \frac{\pi}{2} + \pi$ etc; ie $\theta \neq \pi, -\pi, 3\pi, -3\pi$ etc

Thus, $u = sin\theta$ takes all values in the interval $[-1,1]$

and $v = -cos\theta$ takes all values in the interval $[-1,1)$

and so the locus of w is the circle $u^2 + v^2 = 1$, excluding the point (0,1).

(ii) Using the same substitution as in (i), with $-1 < x < 1$,

$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$ or $-\frac{\pi}{4} + \pi < \frac{\theta}{2} < \frac{\pi}{4} + \pi$ etc

ie $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $-\frac{\pi}{2} + 2\pi < \theta < \frac{\pi}{2} + 2\pi$ etc

Thus, $u = sin\theta$ takes all values in the interval $(-1,1)$

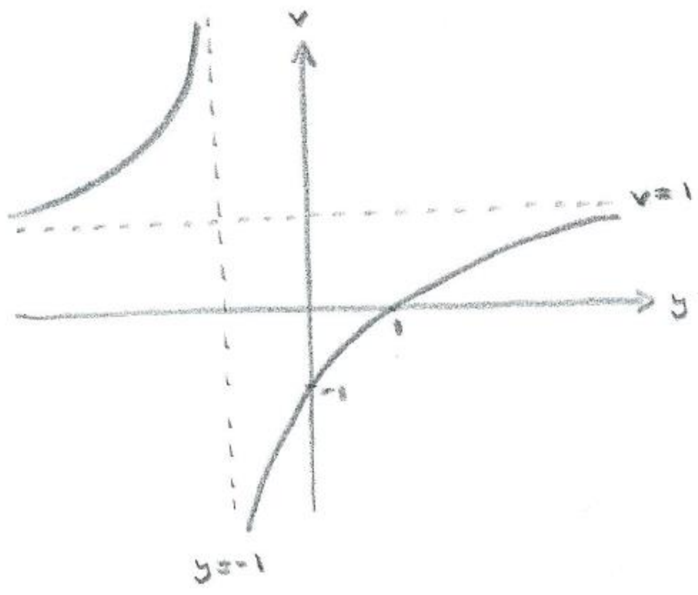
and $v = -cos\theta$ takes all values in the interval $[-1,0)$

and so the locus of w is the part of the circle $u^2 + v^2 = 1$ below the u-axis.

(iii) $x = 0 \Rightarrow u = 0$ & $v = \frac{y^2-1}{(1+y)^2} = \frac{y-1}{y+1}$

Sketching v as a function of y, we get the graph shown below.

[As an alternative to applying the usual methods of curve sketching, we can write $\frac{y-1}{y+1} = 1 - \frac{2}{y+1}$, and so the graph is obtained by transforming $v = \frac{1}{y}$ as follows: translation of $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, followed by stretch of scale factor 2 in the v-direction, followed by



a reflection in the y-axis, and then a translation of $\binom{0}{1}$.]

As $-1 < y < 1, v < 0$, and so the locus of w is the negative imaginary axis.

(iv) $y = 1, -\infty < x < \infty \Rightarrow u = \frac{2x}{x^2+4}$ & $v = \frac{x^2}{x^2+4}$

Let $x = 2tan\left(\frac{\theta}{2}\right)$, so that $u = \frac{4tan\left(\frac{\theta}{2}\right)}{4sec^2\left(\frac{\theta}{2}\right)} = \frac{1}{2}sin\theta$

and $v = \frac{4tan^2\left(\frac{\theta}{2}\right)}{4sec^2\left(\frac{\theta}{2}\right)} = sin^2\left(\frac{\theta}{2}\right)$

As in (i), $-\infty < x < \infty \Rightarrow sin\theta$ takes all values in the interval $[-1,1]$, so that $-\frac{1}{2} \leq u \leq \frac{1}{2}$

As $\frac{\theta}{2}$ can take all values except $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc, $sin\left(\frac{\pi}{2}\right)$ can take all values except ± 1, and hence $0 \leq v < 1$

Also, $u = sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) = \sqrt{v}\sqrt{1-v}$, so that $u^2 = v(1-v)$

$= -(v^2 - v) = -\left(v - \frac{1}{2}\right)^2 + \frac{1}{4}$

and $u^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{4}$

Thus the locus of w is the circle of radius $\frac{1}{2}$ with centre $(0, \frac{1}{2})$, excluding the point $(0, 1)$.