**STEP 2011, Paper 3, Q5 – Solution** (2 pages; 12/6/18)

$r^2 d\theta = r^2 \frac{d\theta}{dt} dt$   (1)

[in order to obtain a $\frac{d\theta}{dt}$ term: ]

Differentiate each of $x = r cos\theta$ & $y = r sin\theta$   (2)  wrt $t$:

$\frac{dx}{dt} = \frac{dr}{dt} cos\theta - r sin\theta \frac{d\theta}{dt}$  (3) &  $\frac{dy}{dt} = \frac{dr}{dt} sin\theta + r cos\theta \frac{d\theta}{dt}$  (4)

[Eliminating the unwanted $\frac{dr}{dt}$ term:]

$sin\theta \times (3) - cos\theta \times (4)$:

$\frac{dx}{dt} sin\theta - \frac{dy}{dt} cos\theta = -r sin^2\theta \frac{d\theta}{dt} - r cos^2\theta \frac{d\theta}{dt}$

Multiplying by $r$, and applying (2):

$\frac{dx}{dt} y - \frac{dy}{dt} x = -r^2 \frac{d\theta}{dt}$

Then, from (1), $r^2 d\theta = r^2 \frac{d\theta}{dt} dt = \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt$ , as required

$A: (x - a cost, y - a sint)$

$B: (x + b cost, y + b sint)$

From (*), $[A] =$

$$\frac{1}{2}\int_0^{2\pi} (x - a cost)(\frac{dy}{dt} - a cost) - (y - a sint)\left(\frac{dx}{dt} + a sint\right) dt$$

$= \frac{1}{2}\int_0^{2\pi} (x\frac{dy}{dt} - y\frac{dx}{dt}) dt$

$+ \frac{1}{2}\int_0^{2\pi} (-x a cost - a cost \frac{dy}{dt} + a^2 cos^2 t$

$-y a sint + a sint \frac{dx}{dt} + a^2 sin^2 t)\ dt$

$= [P] + \frac{1}{2}\int_0^{2\pi} -acost\left(x + \frac{dy}{dt}\right) - asint\left(y - \frac{dx}{dt}\right)dt$

$+ \frac{1}{2}\int_0^{2\pi} a^2 dt$

$= [P] - af + \frac{1}{2}a^2(2\pi)$

$= [P] + \pi a^2 - af$ , as required

From (*), $[B] =$

$$\frac{1}{2}\int_0^{2\pi}(x + bcost)(\frac{dy}{dt} + bcost) - (y + bsint)\left(\frac{dx}{dt} - bsint\right)dt$$

$= \frac{1}{2}\int_0^{2\pi}(x\frac{dy}{dt} - y\frac{dx}{dt})dt$

$+ \frac{1}{2}\int_0^{2\pi}(xbcost + bcost\frac{dy}{dt} + b^2cos^2t$

$+ybsint - bsint\frac{dx}{dt} + b^2sin^2t)\,dt$

$= [P] + \frac{1}{2}\int_0^{2\pi} bcost\left(x + \frac{dy}{dt}\right) + bsint\left(y - \frac{dx}{dt}\right)dt$

$+ \frac{1}{2}\int_0^{2\pi} b^2 dt$

$= [P] + bf + \frac{1}{2}b^2(2\pi)$

$= [P] + \pi b^2 + bf$

The required area is $[A] - [P]$ and $[B] = [A]$

Eliminating $f$ from the expressions for $[A]$ & $[B]$,

$f = \frac{[P] + \pi a^2 - [A]}{a}$ & $f = \frac{[B] - [P] - \pi b^2}{b}$

Hence $b[P] + \pi a^2 b - b[A] = a[B] - a[P] - \pi ab^2$

and $(a + b)[P] - (a + b)[A] = -\pi ab(b + a)$, so that $[A] - [P] = \pi ab$, as required.