STEP 2011, Paper 3, Q2 – Solution (2 pages; 12/6/18)

[If it isn't clear what the "by considering $q^{n-1}f(\frac{p}{q})$" is getting at, a simple quadratic example may help - not that it ends up providing any extra insight in this case, but it can make the problem seem less daunting (and there's less writing to do!)]

$0 = q^{n-1}f\left(\frac{p}{q}\right)$

$= \frac{1}{q}p^n + a_{n-1}p^{n-1} + a_{n-2}qp^{n-2} + \cdots + a_1q^{n-2}p + a_0q^{n-1}$

As p, q & $the\ a_i$ are all integers, the sum of the terms from the 2nd one onwards must be an integer, and hence the term $\frac{1}{q}p^n$ must be an integer (as it equals minus this sum of terms). Thus $q = 1$, as p & q have no common factor greater than 1. Hence any rational root of $f(x) = 0$ must be an integer.

(i) Suppose that the nth root of 2 is rational. Then, if x is the nth root, $x^n - 2 = 0$, and it follows from the result just established that x must be an integer (for $n \geq 2$).

But $x^n = 2$ has no integer solutions, so we have a contradiction, and hence our initial supposition must be incorrect; ie the nth root of 2 must be irrational.

(ii) Suppose that there is a rational root. Then, from the initial result it follows that the root is an integer.

Let $f(x) = x^3 - x + 1$

Considering positive roots, $f(1) = 1$, and we see that $f(n+1) > f(n)$ for $n \geq 1$; thus $f(n) \neq 0$ for these roots.

Considering other roots, $f(0) = 1, f(-1) = 1, f(-2) = -5$ and we see that $f(n-1) < f(n)$ for $n \leq -2$; thus $f(n) \neq 0$ for these roots.

So we have a contradiction, and hence the cubic has no rational roots.

[In the official solutions, mention is made of the fact that "there can only be one real root", though that doesn't seem to be needed for the proof.]

(iii) The same approach can be applied as in (ii).

[The official solutions go to town here to prove something that seems fairly obvious. Had the result been just part of a longer proof, the examiners might not have bothered. It's always hard to tell how much is wanted. The last sentence of the official solutions ("Parts (ii) and (iii) could be shown ...") seems to endorse the above approach though.]

[In the last sentence of the Examiners' Report ("However, considering x being odd or even ..."), presumably they meant n instead of x.]