STEP 2011, Paper 3, Q1 – Solution (2 pages; 12/6/18)

(i) Integrating factor $= exp\{\int -\left(\frac{x+2}{x+1}\right) dx\} = exp\{-\int 1 + \frac{1}{x+1}\ dx\}$

$= exp\{-x - ln(x+1)\} = \frac{e^{-x}}{x+1}$

[the issue of $x + 1\ \leq 0$ is unlikely to be important here: the IF is just a device to create an exact differential]

So, after multiplying by the IF, $\frac{d}{dx}\left(\frac{ue^{-x}}{x+1}\right) = 0$

and hence $u = A(x+1)e^x$

(ii) If $y = ze^{-x}$, $\frac{dy}{dx} = e^{-x}(\frac{dz}{dx} - z)$

& $\frac{d^2y}{dx^2} = e^{-x}(-\frac{dz}{dx} + z + \frac{d^2z}{dx^2} - \frac{dz}{dx})$

Then the DE (*) gives

$e^{-x}\left(\frac{d^2z}{dx^2}(x+1) - 2\frac{dz}{dx}(x+1) + z(x+1) + x\left(\frac{dz}{dx} - z\right) - z\right) = 0$

so that $\frac{d^2z}{dx^2}(x+1) + \frac{dz}{dx}(-x-2) = 0$

ie the 1st order DE for $\frac{dz}{dx}$: $\frac{d}{dx}\left(\frac{dz}{dx}\right) - \left(\frac{x+2}{x+1}\right)\left(\frac{dz}{dx}\right) = 0$

From (i), $\frac{dz}{dx} = A(x+1)e^x$,

so that $z = A\int(x+1)e^x dx$

$= A(xe^x) - A\int e^x dx + A\int e^x dx = Axe^x + B$

So $y = (Axe^x + B)e^{-x} = Ax + Be^{-x}$

(iii) The general solution is the complementary function

$Ax + Be^{-x}$ from (ii) + a particular integral, which is expected to be a quadratic, since $(x+1)^2$ is a quadratic.

Let the PI be $ax^2 + bx + c$ [in fact, because of the term Ax in the CF, we can set $b = 0$]

Substituting the PI into the DE gives

$(x+1)(2a) + x(2ax + b) - (ax^2 + bx + c) = x^2 + 2x + 1$

Equating coeffs of x^2: $2a - a = 1$, so that $a = 1$

Equating coeffs of x: $2 + b - b = 2$

[It would seem that, had the RHS been $x^2 + 3x + 1$, for example, that no polynomial PI would work (you might like to experiment with a cubic, for example).]

Equating constant terms: $2 - c = 1$, so that $c = 1$

Thus the general solution is $y = x^2 + Ax + 1 + Be^{-x}$