STEP 2011, Paper 3, Q11 – Solution (5 pages; 12/6/18)

The fact that the strings are inextensible seems to mean that the disc rises as it rotates (it's slightly worrying that no mention of this is made in the question - the last part of the official solution does imply this though).

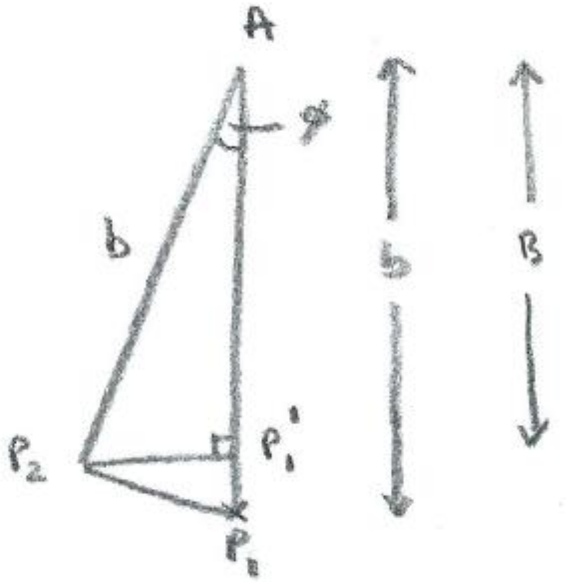

Fig. 1

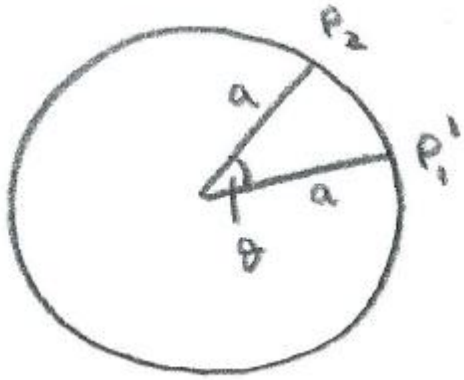

Fig. 2

The 3D configuration of the various points involved in the 1st part is a bit of a nightmare. Let P_1 be the initial position of P; P_2 its position after rotating and rising; and let A be the position of the point on the ceiling vertically above P_1. Then $\phi = \angle P_2AP_1$.

Also, Define P'_1 to be the point vertically above P_1 in the horizontal plane of P_2. Then, from Fig. 1,

$P'_1P_2 = bsin\phi$

and from Fig. 2, $P'_1P_2 = 2(asin\left(\frac{\theta}{2}\right))$,

so that $bsin\phi = 2(asin\left(\frac{\theta}{2}\right))$, as required.

Alternative method

The following is a fairly safe approach involving coordinates (ie it doesn't really involve visualising the actual 3D set-up), but it's obviously much longer.

$A = (a, 0,0)$; $P_1 = (a, 0, -b)$ & $P_2 = (acos\theta, asin\theta, -B)$

(the latter being derived from Fig. 2),

where B is the new vertical distance from the disc to the ceiling.

We also know that $AP_2 = b$, as the strings are inextensible,

so that $(a - acos\theta)^2 + (asin\theta)^2 + B^2 = b^2$ (1)

Then (referring to Fig. 1), by the Cosine rule,

$(P_1P_2)^2 = b^2 + b^2 - 2b^2cos\phi$ (2)

and $(P_1P_2)^2 = (a - acos\theta)^2 + (asin\theta)^2 + (b - B)^2$ (3)

[though the presence of $b - B$ doesn't look very encouraging!]

Writing $X = (a - acos\theta)^2 + (asin\theta)^2$ (4),

we then have, from (1), (2) & (3):

$X + B^2 = b^2$ (5)

$2b^2(1 - cos\phi) = X + (b - B)^2$ (6)

Then (5) & (6) give

$2b^2(1 - cos\phi) = X + b^2 - 2bB + B^2 = 2b^2 - 2bB$

and hence $bcos\phi = B$ (7)

Then, from (5) & (7),

$B^2 = b^2 - X$ & $B^2 = b^2cos^2\phi$, so that $b^2 - X = b^2cos^2\phi$,

and hence $X = b^2sin^2\phi$ (8)

And so, from (4) & (8), $(a - acos\theta)^2 + (asin\theta)^2 = b^2sin^2\phi$

giving $a^2 - 2a^2cos\theta + a^2cos^2\theta + a^2sin^2\theta = b^2sin^2\phi$

$\Rightarrow 2a^2(1 - cos\theta) = b^2sin^2\phi$

Then, as $cos\theta = cos^2\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right) = 1 - 2sin^2\left(\frac{\theta}{2}\right)$,

$4a^2sin^2\left(\frac{\theta}{2}\right) = b^2sin^2\phi$

and $2asin\left(\frac{\theta}{2}\right) = bsin\phi$ (as $\theta < \pi$, so that $sin\left(\frac{\theta}{2}\right) > 0$)

as required.

2nd part

In the new position, if T is the tension in each string, resolving vertically for the forces on the disc:

$nTcos\phi = mg$ (9)

The couple C is defined to be the net moment of forces on the disc in the (new) horizontal plane.

The perpendicular distance from the line of action of the horizontal component of T to the centre of the disc is the perpendicular distance from P'_1P_2 to the centre of the disc;

ie $acos\left(\frac{\theta}{2}\right)$,

so that $C = (nTsin\phi)(acos\left(\frac{\theta}{2}\right))$ (10)

Eliminating nT from (9) & (10) gives

$$C = \left(\frac{mg}{cos\phi}\right) asin\phi cos\left(\frac{\theta}{2}\right) = \frac{mga\left(\frac{2asin\left(\frac{\theta}{2}\right)}{b}\right)cos\left(\frac{\theta}{2}\right)}{\sqrt{1-\left(\frac{2asin\left(\frac{\theta}{2}\right)}{b}\right)^2}} = \frac{mga^2 sin\theta}{\sqrt{b^2-4a^2sin^2\left(\frac{\theta}{2}\right)}}$$

as required.

[The term 'couple' is a bit of a misnomer: it suggests that there are a couple of forces involved, when in fact it applies to any situation where the forces are balanced, but there is not rotational equilibrium.]

last part

As an alternative to using conservation of energy (as in the official solution), we can use the fact that

rate of change of angular momentum = total moment of forces

Let $\Omega(\alpha)$ be the angular velocity, where α is the angle turned by the disc from the vertical (so that $\alpha = \theta$ when the disc is released, and $\alpha = 0$ when the strings are vertical) [noting that α is a variable, whilst θ is a constant], we have

$\frac{d}{dt}(-I\Omega) = C$, where the moment of inertia I of the disc about its axis is $\frac{1}{2}ma^2$ (the negative sign is needed, as C acts in the direction of α decreasing), and

$-\frac{1}{2}ma^2\frac{d\Omega}{d\alpha}\frac{d\alpha}{dt} = \frac{mga^2 sin\alpha}{f(\alpha)}$, where $f(\alpha) = \sqrt{b^2 - 4a^2sin^2\left(\frac{\alpha}{2}\right)}$

Then, as $\frac{d\alpha}{dt} = \Omega$, $-\Omega\frac{d\Omega}{d\alpha} = \frac{2gsin\alpha}{f(\alpha)}$

and $-\frac{1}{2}\Omega^2 = 2gsin\alpha\int\frac{1}{f(\alpha)}d\alpha$

and as $\frac{d}{d\alpha}\left(b^2 - 4a^2sin^2\left(\frac{\alpha}{2}\right)\right) = -4a^2.2sin\left(\frac{\alpha}{2}\right)\left(\frac{1}{2}\right)cos\left(\frac{\alpha}{2}\right)$

$= -2a^2 sin\alpha$,

$-\frac{1}{2}\Omega^2 = \frac{2g}{(-2a^2)}(-2a^2 sin\alpha)\int \frac{1}{f(\alpha)} d\alpha = -\frac{g}{a^2}\frac{f(\alpha)}{\left(\frac{1}{2}\right)} + C$

so that $\frac{a^2\Omega^2}{4g} = f(\alpha) - C$

and when $\alpha = \theta, \Omega = 0$, so that $C = f(\theta)$

Then $\frac{a^2(\Omega(0))^2}{4g} = f(0) - f(\theta) = b - f(\theta)$

The angular speed required ω is $-\Omega(0)$

and thus $\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 sin^2\left(\frac{\theta}{2}\right)}$, as required.