STEP 2011, Paper 2, Q6 – Solution (2 pages; 12/6/18)

[There are two likely methods that this question could be relating to: substitution and Parts. The former doesn't appear to lead anywhere.]

$I = \int f'(x)\{ f'(x)[f(x)]^n\}dx$

By Parts, $I = f'(x)\frac{[f(x)]^{n+1}}{n+1} - \int f''(x)\frac{[f(x)]^{n+1}}{n+1}dx$

$= f'(x)\frac{[f(x)]^{n+1}}{n+1} - \int kf'(x)\frac{[f(x)]^{n+2}}{n+1}dx$

$= f'(x)\frac{[f(x)]^{n+1}}{n+1} - \frac{k}{(n+1)}\frac{[f(x)]^{n+3}}{(n+3)} + C \quad (1)$

(i) [It isn't entirely clear here whether the 'verification' required just amounts to showing that $f''(x) = xf(x)f'(x)$. It seems a bit strange to have to prove something twice!]

For $f(x) = tanx,\ f'(x) = sec^2x\ \ \&\ \ f''(x) = 2secx(secxtanx)$

so that $f''(x) = 2tanxsec^2x = 2f(x)f'(x)$; ie the condition holds with $k = 2$

And $\quad I = \int[f'(x)]^2[f(x)]^n dx = \int(sec^2x)^2tan^nx\ dx \quad (2)$

Then $f'(x)\frac{[f(x)]^{n+1}}{n+1} - \frac{k}{(n+1)}\frac{[f(x)]^{n+3}}{(n+3)}$

$= sec^2x\frac{tan^{n+1}x}{n+1} - \frac{2}{(n+1)}\frac{tan^{n+3}x}{(n+3)}$

$= \frac{sin^{n+1}x}{(n+1)cos^{n+3}x} - \frac{2tan^{n+3}x}{(n+1)(n+3)}$

Differentiating, we get

$$\left(\frac{1}{n+1}\right)\frac{(cos^{n+3}x)(n+1)(sin^nx)(cosx) - (sin^{n+1}x)(n+3)(cos^{n+2})(-sinx)}{cos^{2n+6}x}$$

$-\frac{2(n+3)tan^{n+2}x(sec^2x)}{(n+1)(n+3)}$

$= \left(\frac{1}{n+1}\right)\frac{(cos^2x)(n+1)(sin^nx)+(sin^{n+2}x)(n+3)}{(cos^{n+4}x)} - \frac{2(sin^{n+2}x)}{(n+1)(cos^{n+4}x)}$

$= \frac{(sin^nx)\{(cos^2x)(n+1)+(sin^2x)(n+3)-2sin^2x\}}{(n+1)(cos^{n+4}x)}$

$= \frac{(sin^nx)(n+1)}{(n+1)(cos^{n+4}x)}$

[Fortunately! On reflection, using the product rule on $sec^2x\,\frac{tan^{n+1}x}{n+1}$ would probably have been simpler.]

$= (sec^2x)^2tan^nx$, agreeing with (2)

$\int \frac{sin^4x}{cos^8x}dx = \int (sec^2x)^2tan^4x\,dx$

$= sec^2x\frac{tan^5x}{5} - \frac{2}{5}\frac{tan^7x}{(7)} + C$ (from (1))

$= \frac{1}{5}sec^2xtan^5x - \frac{2tan^7x}{35} + C$

(ii) Suppose that $f(x) = secx + tanx$

[The examiners don't usually try to trap students: it's always worth investigating the obvious possibility.]

Then $f'(x) = secxtanx + sec^2x = secx(tanx + secx)$

and $f''(x) = secxtanx(tanx + secx) + secx(sec^2x + secxtanx)$

$= (secx + tanx)(secxtanx + sec^2x) = f(x)f'(x)$

Then $sec^2x(secx + tanx)^6 = (f'(x))^2[f(x)]^4$

Thus the necessary condition applies for the initial result to be used (with $k = 1$ and $n = 4$).

So $\int sec^2x(secx + tanx)^6dx$

$= \int [secx(secx + tanx)]^2(secx + tanx)^4dx$

$= \frac{1}{5}\ secx(tanx + secx)(secx + tanx)^5 - \frac{1}{5(7)}(secx + tanx)^7 + C$

from (1)

$= \frac{1}{35}(secx + tanx)^6\{7secx - (secx + tanx)\} + C$

$= \frac{1}{35}(secx + tanx)^6(6secx - tanx) + C$