STEP 2011, Paper 2, Q2 – Solution (4 pages; 31/5/18)

1st 10 cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000

[no marks for this, apparently!]

(i) [Note: In our search for a solution of (*), we are restricting ourselves to cases where $x + y = k$]

$kz^3 = x^3 + y^3 = x^3 + (k - x)^3$

$= x^3 + k^3 - 3k^2x + 3kx^2 - x^3$

$= k^3 - 3k^2x + 3kx^2$

$\Rightarrow z^3 = k^2 - 3kx + 3x^2$, as required. (1)

$\frac{4z^3 - k^2}{3} = \frac{3k^2 - 12kx + 12x^2}{3} = k^2 - 4kx + 4x^2$

$= (k - 2x)^2$, which is a perfect square, as k & x are integers.

As $\frac{4z^3 - k^2}{3}$ is a perfect square, $4z^3 - k^2 \geq 0$,

so that $z^3 \geq \frac{1}{4}k^2$

To show that $z^3 < k^2$, from (1) we need to show that

$-3kx + 3x^2 < 0 \Leftrightarrow x(x - k) < 0$

As $x > 0$ & $x < k$, this is true.

So $\frac{1}{4}20^2 \leq z^3 < 20^2$; ie $100 \leq z^3 < 400$

and hence z must be 5, 6 or 7 (if there is to be a sol'n with $x + y = 20$).

From (1), $z = 5 \Rightarrow 125 = 400 - 60x + 3x^2$

$\Rightarrow 3x^2 - 60x + 275 = 0$, which doesn't appear to factorise (and in fact has a negative discriminant)

$z = 6 \Rightarrow 3x^2 - 60x + 184 = 0$, which again doesn't appear to factorise (although the discriminant is positive)

$z = 7 \Rightarrow 3x^2 - 60x + 57 = 0$

$\Rightarrow (3x - 57)(x - 1) = 0$

$\Rightarrow x = 1, y = 19$

Notes:

(a) With hindsight, there is more likely to be a solution for larger values of z.

(b) Alternatively, $x^3 + y^3$ can be factorised as

$(x + y)(x^2 - xy + y^2)$, and the $x + y$ then cancels out when we set $x + y = k$ in $x^3 + y^3 = kz^3$

(c) To find the correct value of z, the official sol'n uses the fact that $\frac{4z^3 - k^2}{3}$ must be a perfect square.

(ii) [Note that we won't be setting $x + y = k$ here, as that would mean that $z^2 = 19$. So it seems that the result $\frac{1}{4}k^2 \leq z^3 < k^2$ won't be relevant here. Of course it's possible that we are expected to come up with a similar inequality, so we should be on the lookout for perfect squares. However, the only thing that definitely seems to be worthwhile is the following:]

From $x^3 + y^3 = 19z^3$ and $x + y = z^2$, we can eliminate y, to give

$x^3 + (z^2 - x)^3 = 19z^3$

$\Rightarrow z^6 - 3z^4x + 3z^2x^2 = 19z^3$

$\Rightarrow 3x^2 - 3z^2x + z^4 - 19z = 0 \quad (2)$

[bearing in mind that quadratics are virtually the only type of equation that can be tackled easily]

For this to have a solution, the discriminant must be non-negative;

ie $9z^4 - 12(z^4 - 19z) \geq 0$

$\Rightarrow -3z^4 + 12(19)z \geq 0$

$\Rightarrow z^3 - 76 \leq 0$ (as $z > 0$)

$\Rightarrow z = 1, 2, 3 \; or \; 4$

When $z = 4$, (2) $\Rightarrow 3x^2 - 48x + 256 - 76 = 0$,

so that $x^2 - 16x + 60 = 0$

$\Rightarrow (x - 6)(x - 10) = 0 \Rightarrow x = 6, y = z^2 - x = 10$

(noting that $x < y$ [it's a good idea to re-read the question at this point])

When $z = 3$, (2) $\Rightarrow 3x^2 - 27x + 81 - 57 = 0$,

so that $x^2 - 9x + 8 = 0$

$\Rightarrow (x - 1)(x - 8) = 0 \Rightarrow x = 1, y = z^2 - x = 8$

(and we have found two solutions, as requested).

Note

In fact, in turns out that the examiners **were** expecting a similar inequality to that in (i). But the appropriate perfect square only emerges by requiring the discriminant of the quadratic in x to be a perfect square (the method the examiners used in (i)).