STEP 2011, Paper 1, Q2 – Solution (2 pages; 11/6/18)

$\int_0^1 \frac{xe^x}{1+x}\ dx = \int_0^1 \frac{xe^x+e^x}{1+x} - \frac{e^x}{1+x}\ dx$

$= \int_0^1 e^x\ dx - E = [e^x]_0^1 - E = e - 1 - E$

$\int_0^1 \frac{x^2e^x}{1+x}\ dx = \int_0^1 \frac{x^2e^x+xe^x}{1+x} - \frac{xe^x}{1+x}\ dx$

$= \int_0^1 xe^x\ dx - (e - 1 - E)$

$= [xe^x]_0^1 - \int_0^1 e^x\ dx\ - e + 1 + E$

$= e - (e - 1) - e + 1 + E$

$= E + 2 - e$

(i) Let $u = \frac{1-x}{1+x}$ [which will hopefully tidy up the integral a bit]

so that $du = \frac{(1+x)(-1)-(1-x)}{(1+x)^2}\ dx = -\frac{2}{(1+x)^2}\ dx$

and $I = \int_0^1 \frac{e^{\frac{1-x}{1+x}}}{1+x}\ dx = \int_1^0 e^u[-\frac{(1+x)}{2}]du$

Now $u(1+x) = 1 - x,$ so that $x(u+1) = 1 - u$

and hence $1 + x = 1 + \frac{1-u}{u+1} = \frac{2}{u+1}$

Then $I = -\frac{1}{2}\int_1^0 e^u\left(\frac{2}{1+u}\right) du = \int_0^1 \frac{e^u}{1+u}\ du = E$

(ii) Let $u = x^2$, so that $du = 2x\ dx$

[The obvious approach usually works with STEP!]

Then $I = \int_1^{\sqrt{2}} \frac{e^{x^2}}{x}\, dx = \int_1^2 e^u \left(\frac{1}{2u}\right) du$

Now let $z = u - 1$

[so that the limits become 0 & 1, and also $\frac{1}{u} = \frac{1}{1+z}$]

so that $I = \frac{1}{2}\int_0^1 \frac{e^{z+1}}{1+z}\, dz = \frac{eE}{2}$