STEP 2010, Paper 3, Q2 – Solution (2 pages; 10/6/18)

(i) $cosha = \frac{1}{2}(e^a + e^{-a})$ [a fairly obvious clue that this is to be used in the next part]

As an alternative to factorising $x^2 + x(e^a + e^{-a}) + 1$ as

$(x + e^a)(x + e^{-a})$,

consider $(x + cosha)^2 = x^2 + 2xcosha + cosh^2a$

Thus $x^2 + 2xcosha + 1 = (x + cosha)^2 - cosh^2a + 1$

$= (x + cosha)^2 - sinh^2a$

Then $\frac{1}{x^2+2xcosha+1} = \frac{1}{2sinha}(\frac{1}{x+cosha-sinha} - \frac{1}{x+cosha+sinha})$

and $\int_0^1 \frac{1}{x^2+2xcosha+1}dx$

$$= \frac{1}{2sinha}[ln(x + cosha - sinha) - ln(x + cosha + sinha)]_0^1$$

$= \frac{1}{2sinha}\{ln\left(\frac{1+cosha-sinha}{1+cosha+sinha}\right) - ln\left(\frac{cosha-sinha}{cosha+sinha}\right)\}$

$= \frac{1}{2sinha}ln\left(\frac{(1+cosha-sinha)(cosha+sinha)}{(1+cosha+sinha)(cosha-sinha)}\right)$

$= \frac{1}{2sinha}ln\left(\frac{(1+e^{-a})e^a}{(1+e^a)e^{-a}}\right) = \frac{1}{2sinha}ln\left(\frac{e^a+1}{(1+e^a)e^{-a}}\right) = \frac{1}{2sinha}ln(e^a)$

$= \frac{a}{2sinha}$, as required.

(ii) $x^2 + 2xsinha - 1 = (x + sinha)^2 - sinh^2a - 1$

$(x + sinha)^2 - cosh^2a$

Then the integrand is the same as for (i), but with $cosha$ & $sinha$ reversed,

so that $\int_1^\infty \frac{1}{x^2+2xsinha-1}dx$

$$= \frac{1}{2cosha}[ln(x + sinha - cosha) - ln(x + cosha + sinha)]_1^\infty$$

$$= \frac{1}{2cosha}\{ln(1) - ln\left(\frac{1+sinha-cosha}{1+cosha+sinha}\right)\}$$

$$= \frac{1}{2cosha}ln\left(\frac{1+e^a}{1-e^{-a}}\right) = \frac{1}{2cosha}ln\left(\frac{(e^{-\frac{a}{2}}+e^{\frac{a}{2}})e^{\frac{a}{2}}}{(e^{\frac{a}{2}}-e^{-\frac{a}{2}})e^{-\frac{a}{2}}}\right)$$

$$=\frac{1}{2cosha}ln\left(\frac{e^a cosh(\frac{a}{2})}{sinh(\frac{a}{2})}\right) = \frac{1}{2cosha}(a + ln\left(coth\left(\frac{a}{2}\right)\right))$$

For $\int_0^\infty \frac{1}{x^4+2x^2cosha+1}dx$, the x in (i) is being replaced with x^2, so that we have

$$\frac{1}{2sinha}\int_0^\infty\left(\frac{1}{x^2+cosha-sinha} - \frac{1}{x^2+cosha+sinha}\right)dx$$

$$= \frac{1}{2sinha}\int_0^\infty\left(\frac{1}{x^2+e^{-a}} - \frac{1}{x^2+e^{a}}\right)dx$$

$$= \frac{1}{2sinha}\left[\frac{1}{e^{-a/2}}tan^{-1}\left(\frac{x}{e^{-a/2}}\right) - \frac{1}{e^{a/2}}tan^{-1}\left(\frac{x}{e^{a/2}}\right)\right]_0^\infty$$

$$= \frac{1}{2sinha}\left(\frac{\pi}{2}e^{\frac{a}{2}} - \frac{\pi}{2}e^{-\frac{a}{2}} - 0\right)$$

$$= \frac{\pi}{2sinha}sinh\left(\frac{a}{2}\right) = \frac{\pi}{4cosh(\frac{a}{2})}$$