

STEP 2010, Paper 2, Q4 – Solution (3 pages; 5/4/25)

4 (i) Let

$$I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx.$$

Use a substitution to show that

$$I = \int_0^a \frac{f(a-x)}{f(x) + f(a-x)} dx$$

and hence evaluate I in terms of a .

Use this result to evaluate the integrals

$$\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)} dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(x + \frac{\pi}{4})} dx.$$

(ii) Evaluate

$$\int_{\frac{1}{2}}^2 \frac{\sin x}{x(\sin x + \sin \frac{1}{x})} dx.$$

Introduction

The first part of the question regularly involves something very obvious (though in this case, a slight modification is needed!)

Having established the initial result, $\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)} dx$ is most likely to be a straightforward application of it.

In the case of definite integrals, the limits can be very revealing.

Note that $\int_a^b = -\int_b^a$

For the subsequent integrals, there is bound to be some complication. Possibilities include:

(a) Some manipulation may be needed in order to be able to apply the result.

(b) Generalising the earlier result in some way [eg if it had involved $\int \sin x \cos^3 x dx$, this could be generalised to $\int \sin x \cos^n x dx$]

(c) Deriving a new result by applying a similar idea.

Try observing any interesting features of the question. Does anything stand out in the case of $\int_{1/2}^2 \frac{\sin x}{x(\sin x + \sin(\frac{1}{x}))} dx$?

Solution

(i) Let $x = a - u$ [the u will then be replaced with x , to give $f(a - x)$ in the numerator], so that

$I = \int_a^0 \frac{-f(a-u)}{f(a-u)+f(u)} du = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)} dx$, giving the required result.

Then $2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)} dx = \int_0^a dx = a$, so that $I = \frac{a}{2}$

$$I_1 = \int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)} dx = \int_0^1 \frac{\ln(x+1)}{\ln[(2-x)(1+x)]} dx = \int_0^1 \frac{\ln(x+1)}{\ln(2-x) + \ln(1+x)} dx$$

Using the previous result, a will be 1, and $\ln(x+1)$ will be $f(x)$, making $f(a-x) = \ln([1-x]+1) = \ln(2-x)$,

so that $I_1 = 1/2$

$$\begin{aligned} I_2 &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(x+\frac{\pi}{4})} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} dx = \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \sin(\frac{\pi}{2}-x)} dx \\ &= \sqrt{2} \left(\frac{\frac{\pi}{2}}{2} \right) = \frac{\pi\sqrt{2}}{4} \end{aligned}$$

(ii) In the case of $\int_{1/2}^2 \frac{\sin x}{x(\sin x + \sin(\frac{1}{x}))} dx$, the bottom limit isn't 0, and

there isn't an obvious substitution that produces 0 ($u = x - \frac{1}{2}$ doesn't look very promising, for example), but it may be possible to apply a similar idea to that used to produce the original result.

Let $u = 1/x$, so that $du = -(1/x^2)dx$, so that $\frac{dx}{x} = -\frac{du}{u}$

$$\begin{aligned} \text{Then } I_3 &= \int_{1/2}^2 \frac{\sin x}{x(\sin x + \sin(\frac{1}{x}))} dx = \int_2^{1/2} \frac{-\sin(\frac{1}{u})}{u(\sin(\frac{1}{u}) + \sin u)} du = \\ &\int_{1/2}^2 \frac{\sin(\frac{1}{x})}{x(\sin(\frac{1}{x}) + \sin x)} dx \end{aligned}$$

$$\begin{aligned} \text{Hence } 2I_3 &= \int_{\frac{1}{2}}^2 \frac{\sin x + \sin(\frac{1}{x})}{x(\sin x + \sin(\frac{1}{x}))} dx = \int_{\frac{1}{2}}^2 \frac{1}{x} dx = \ln 2 - \ln\left(\frac{1}{2}\right) \\ &= \ln 2 + \ln 2 = 2\ln 2, \text{ so that } I_3 = \ln 2 \end{aligned}$$