STEP 2010, Paper 2, Q13 – Solution (3 pages; 9/6/18)

(i) P(success|PPQ) $= [1 - P(loses\ twice\ to\ P)]P(beats\ Q)$

$= [1-(1-p)^2]q = (2p-p^2)q = pq(2-p)$

P(success|PQQ) $= qp(2-q)$ (reversing the roles of P & Q)

As $p < q, -p > -q$; $2-p > 2-q$ & $hence$

P(success|PPQ) $>$ P(success|PQQ), as required

(ii) P(success|PPPQ) $= [1-(1-p)^3]q = [p^3 - 3p^2 + 3p]q$

$= pq(p^2 - 3p + 3)$

Similarly, P(success|PPPQ) $= qp(q^2 - 3q + 3)$

And P(success|PPQQ) $= [1-(1-p)^2][1-(1-q)^2]$

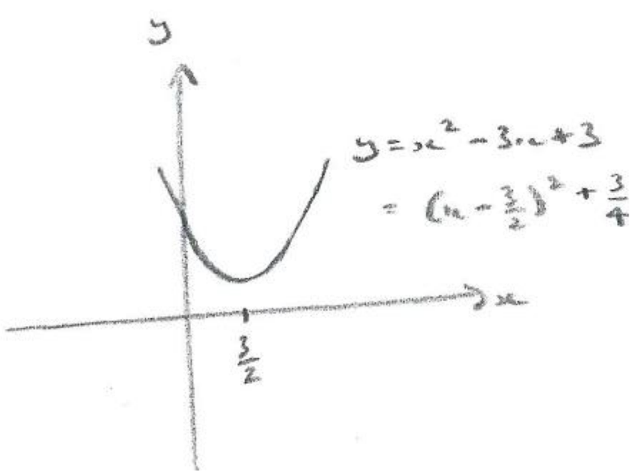
$= (2p-p^2)(2q-q^2) = pq(2-p)(2-q)$

When $q - p > \frac{1}{2}$, rtp (result(s) to prove):

$p^2 - 3p + 3 > q^2 - 3q + 3$ (A)

and $p^2 - 3p + 3 > (2-p)(2-q)$ (B)

For (A), consider the graph of $f(x) = x^2 - 3x + 3$:



The x-coord. of the minimum is $\frac{3}{2}$ (being the same as that of

$f(x) = x^2 - 3x = x(x-3)$; ie halfway between the roots; or by completing the square). So, for $p < q < 1 < \frac{3}{2}$, $f(p) > f(q)$, as required.

[Note that we didn't use the fact that $q - p > \frac{1}{2}$; ie strategy 1 is always better than strategy 3]

For (B), we want to show that $p^2 - 3p + 3 - (2-p)(2-q) > 0$;

ie $p^2 - p - 1 + 2q - pq > 0$ (where $q - p > \frac{1}{2}$; ie $q > \frac{1}{2} + p$)

$LHS > p^2 - p - 1 + (\frac{1}{2} + p)(2-p)$, as $p < 2$ and hence $2 - p > 0$

So $LHS > \frac{p}{2} > 0$, as required

When $q - p < \frac{1}{2}$, we want to find examples for which

$A(say) = p^2 - 3p + 3 - (2-p)(2-q)$ is (a) +ve, and (b) -ve

Let $q = p + \frac{1}{2} - \delta$ (where $\delta > 0$)

Then $2 - q = \frac{3}{2} - p + \delta$

and $A = p^2 - 3p + 3 - (2-p)(\frac{3}{2} - p + \delta)$

$= \frac{p}{2} + \delta(p-2) = B$, say

Noting that $\delta < 1/2$, in order that $p < q$,

consider $\delta = 1/4$, so that $q = p + 1/4$

Then $B = \frac{3p}{4} - \frac{1}{2}$

For (a), we want $B > 0$, so that $\frac{3p}{4} - \frac{1}{2} > 0$ and $p > \frac{2}{3}$

Noting that $q = p + 1/4$, we could let $p = \frac{17}{24}$, so that $q = \frac{23}{24}$

For (b), we want $p < \frac{2}{3}$; eg $p = \frac{15}{24}$, so that $q = \frac{21}{24}$