STEP 2010, Paper 1, Q3 – Solution (3 pages; 8/6/18)

[The points P, Q, R & S can be seen to be points on an ellipse and it is tempting to consider a geometrical proof. However, this would be unknown territory and the factor formulae in the 1st part of the question are obviously intended to be used.]

$\sin(x+y) - \sin(x-y)$

$= sinxcosy + cosxsiny - (sinxcosy - cosxsiny)$

$= 2cosxsiny$, as required.

Let $A = x + y$ & $B = x - y$

Then $x = \frac{1}{2}(A+B)$ & $y = \frac{1}{2}(A-B)$

giving $sinA - sinB = 2\cos[\frac{1}{2}(A+B)]\sin[\frac{1}{2}(A-B)]$, as required.

For $cosA - cosB$, consider $\cos(x+y) - \cos(x-y)$

$= cosxcosy - sinxsiny - [cosxcosy + sinxsiny]$

$= -2sinxsiny$

Then, with $A = x + y$ & $B = x - y$ again,

$cosA - cosB = -2sin[\frac{1}{2}(A+B)]\sin[\frac{1}{2}(A-B)]$, as required.

[For "if and only if" (or "necessary and sufficient") proofs, it is very important to show that the logic works in both directions. However it seems that the examiners will normally be satisfied by use of the ⇔ ("implies and is implied by") symbol - provided that, at each point in the chain, it is fairly clear that the two statements are equivalent.]

PQ & SR parallel (and not vertical) $\Leftrightarrow$ the gradients of PQ & SR are equal

$\Leftrightarrow \frac{bsinq-bsinp}{acosq-acosp} = \frac{bsinr-bsins}{acosr-acoss}$

$\Leftrightarrow (sinq - sinp)(cosr - coss)$

$= (sinr - sins)(cosq - cosp) \quad (1)$

since $r \neq s$ and r & s don't differ by a multiple of 2π (as they both lie in the range $[0,2\pi)$, and hence $cosr - coss \neq 0$

(and similarly for p & q)

Then, applying the results for $sinA - sinB$ & $cosA - cosB$,

(1) $\Leftrightarrow$

$2\cos\left[\frac{1}{2}(q+p)\right]\sin\left[\frac{1}{2}(q-p)\right](-2)\sin\left[\frac{1}{2}(r+s)\right]\sin[\frac{1}{2}(r-s)]$

$= 2\cos\left[\frac{1}{2}(r+s)\right]\sin\left[\frac{1}{2}(r-s)\right](-2)\sin\left[\frac{1}{2}(q+p)\right]\sin[\frac{1}{2}(q-p)]$

$\Leftrightarrow \cos\left[\frac{1}{2}(q+p)\right]\sin\left[\frac{1}{2}(r+s)\right] = \cos\left[\frac{1}{2}(r+s)\right]\sin\left[\frac{1}{2}(q+p)\right] \ (2)$

Then, applying the result for $2cosxsiny$,

(2) $\Leftrightarrow \sin\left[\frac{1}{2}(q+p+r+s)\right] - \sin\left[\frac{1}{2}(q+p-r-s)\right]$

$= \sin\left[\frac{1}{2}(r+s+q+p)\right] - \sin[\frac{1}{2}(r+s-q-p)]$

$\Leftrightarrow \sin\left[\frac{1}{2}(q+p-r-s)\right] = \sin[\frac{1}{2}(r+s-q-p)]$

$-\sin[\frac{1}{2}(q+p-r-s)]$

$\Leftrightarrow q+p-r-s = 2n\pi$, where $n \in \mathbb{Z}$ (3)

As $r + s > q + p$, we need only consider $r + s - p - q = 2k\pi$, where $k \in \mathbb{Z}^+$

Now $r + s \neq q + p + 4\pi$, since p & q are both positive and r & s are both $< 2\pi$. Thus $r + s - q - p \neq 4\pi$, and similarly for higher multiples of 2π.

Thus (3) $\Leftrightarrow r + s - p - q = 2\pi$, as required.