STEP 2010, Paper 1, Q1 – Solution (2 pages; 8/6/18)

Equating coeffs of x^2: $5 = a + bc^2$ (1)

of y^2: $2 = a + b$ (2)

of xy: $-6 = -2a + 2bc$ (3)

of x: $4 = 4a$ (4)

of y: $-4 = -4a$ (duplicates (4))

constant term: $0 = 4a + d$ (5)

Then (4) $\Rightarrow a = 1$

(5) $\Rightarrow d = -4$

(2) $\Rightarrow b = 1$

(1) $\Rightarrow c = \pm 2$

(3) $\Rightarrow -6 = -2 + 2c \Rightarrow c = -2$

So $\boldsymbol{a = 1, b = 1, c = -2}$ & $\boldsymbol{d = -4}$

Suppose that

$6x^2 + 3y^2 - 8xy + 8x - 8y$

$= A(x - y + 2)^2 + B(-2x + y)^2 + D$

Then, equating coeffs as before:

$6 = A + 4B$ (6)

$3 = A + B$ (7)

$-8 = -2A - 4B$ (8)

$8 = 4A$ (9)

$-8 = -4A$ (which duplicates (9))

$0 = 4A + D$ (10)

Then (6) & (7) $\Rightarrow 3 = 3B \Rightarrow B = 1, A = 2$,

which (fortunately) agrees with (8) & (9)

And (10) then $\Rightarrow D = -8$

Thus the simultaneous eq'ns become:

$(x - y + 2)^2 + (-2x + y)^2 - 4 = 9$

& $2(x - y + 2)^2 + (-2x + y)^2 - 8 = 14$

giving $p + q = 13$ & $2p + q = 32$,

where $p = (x - y + 2)^2$ & $q = (-2x + y)^2$

and thus $p = 9, q = 4$,

so that $x - y + 2 = \pm 3$ & $-2x + y = \pm 2$

Hence each of the following cases leads to a solution:

(a) $x - y = 1, -2x + y = 2 \Rightarrow -x = 3 \Rightarrow x = -3, y = -4$

(b) $x - y = 1, -2x + y = -2 \Rightarrow -x = -1 \Rightarrow x = 1, y = 0$

(c) $x - y = -5, -2x + y = 2 \Rightarrow -x = -3 \Rightarrow x = 3, y = 8$

(d) $x - y = -5, -2x + y = -2 \Rightarrow -x = -7 \Rightarrow x = 7, y = 12$