STEP 2009, Paper 3, Q8 - Solution (2 pages; 7/6/18)

(i) [Investigating $t = lnx$ leads to $x \to \infty$ as $t \to \infty$]

Let $t = -lnx$, so that $x \to 0$ as $t \to \infty$

Then $\lim_{x\to 0} x^m(lnx)^n = \lim_{t\to\infty} e^{-mt}(-t)^n = (-1)^n \lim_{t\to\infty} e^{-mt}t^n = 0$

$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{xlnx}$ and $\lim_{x\to 0} xlnx = 0$, on setting $m = n = 1$ in the previous result

So $\lim_{x\to 0} e^{xlnx} = e^{\lim_{x\to 0} xlnx} = e^0 = 1$

Thus $\lim_{x\to 0} x^x = 1$

(ii) $I_n = \int_0^1 x^m(lnx)^n dx$

By Parts, $I_{n+1} = \int_0^1 x^m(lnx)^{n+1}dx = \lim_{a\to 0}[\frac{1}{(m+1)}x^{m+1}(lnx)^{n+1}]_a^1$

$-\int_0^1 \frac{1}{(m+1)}x^{m+1}(n+1)(lnx)^n\left(\frac{1}{x}\right)dx$

$= (0-0)$[by the 1st result] $-\frac{n+1}{m+1}\int_0^1 x^m(lnx)^x\ dx$

$= -\frac{(n+1)}{(m+1)}I_n$

Then $I_n = \left(-\frac{n}{m+1}\right)\left(-\frac{(n-1)}{(m+1)}\right)\dots\left(-\frac{1}{m+1}\right)I_0$

$I_0 = \int_0^1 x^m dx = \left[\frac{1}{m+1}x^{m+1}\right]_0^1 = \frac{1}{m+1}$

Hence $I_n = \frac{(-1)^n n!}{(m+1)^{n+1}}$

(iii) $\int_0^1 x^x dx = \int_0^1 e^{xlnx} dx$

[In order to see how to proceed from here, we can compare

$-\left(\frac{1}{2}\right)^2$ and $\left(\frac{1}{3}\right)^3$ with $\frac{(-1)^n n!}{(m+1)^{n+1}}$

$m = 1$ & $n = 1$ $gives -\left(\frac{1}{2}\right)^2$

and $m = 2$ & $n = 2$ $gives$ $\left(\frac{1}{3}\right)^3$ $2!$

This pattern works for the other terms as well.]

Let $f(n) = \frac{(-1)^n n!}{(m+1)^{n+1}}$

Then we want to show that $I = \int_0^1 e^{xlnx} dx = \sum_{n=0}^{\infty} \frac{f(n)}{n!}$

Now $\sum_{n=0}^{\infty} \frac{f(n)}{n!} = \sum_{n=0}^{\infty} \frac{I_n}{n!}$, with $m = n$

$= \sum_{n=0}^{\infty} \int_0^1 \frac{x^n (lnx)^n}{n!} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(xlnx)^n}{n!} \; dx = \int_0^1 e^{xlnx} dx,$

as required

[It's worth checking that we haven't missed something though, as the result $\lim_{x \to 0} x^x = 1$ hasn't been used.]