STEP 2009, Paper 3, Q6 - Solution (5 pages; 27/3/25)

Show that $\left|e^{i\beta} - e^{i\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$\left|e^{i\alpha} - e^{i\beta}\right| \left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right| \left|e^{i\alpha} - e^{i\delta}\right| = \left|e^{i\alpha} - e^{i\gamma}\right| \left|e^{i\beta} - e^{i\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

1st Part

[Aiming to write $e^{i\beta} - e^{i\alpha}$ in the form $e^{ia}(e^{ib} - e^{-ib})$, and noting that the indices initially sum to $i(\beta + \alpha)$, but need to sum to $i(b - b) = 0$, so that we need to reduce the sum by $i(\beta + \alpha)$, and this reduction is to be shared equally between the 2 parts; ie we need to make $ia = \frac{1}{2}i(\beta + \alpha)$]

$e^{i\beta} - e^{i\alpha} = e^{\frac{1}{2}i(\beta+\alpha)}(e^{i\beta-\frac{1}{2}i(\beta+\alpha)} - e^{i\alpha-\frac{1}{2}i(\beta+\alpha)})$

$= e^{\frac{1}{2}i(\beta+\alpha)}(e^{\frac{1}{2}i(\beta-\alpha)} - e^{-\frac{1}{2}i(\beta-\alpha)})$

$= e^{\frac{1}{2}i(\beta+\alpha)}.2\sin(\frac{1}{2}(\beta - \alpha))$,

and so $\left|e^{i\beta} - e^{i\alpha}\right| = |e^{\frac{1}{2}i(\beta+\alpha)}|.2\sin(\frac{1}{2}(\beta - \alpha))$

$= 2\sin(\frac{1}{2}(\beta - \alpha))$

(which is positive when $0 < \alpha < \beta < 2\pi$), as required.

Alternative Method

$\left|e^{i\beta} - e^{i\alpha}\right| = |cos\beta + isin\beta - (cos\alpha + isin\alpha)|$

So $\left|e^{i\beta} - e^{i\alpha}\right|^2 = (cos\beta - cos\alpha)^2 + (sin\beta - sin\alpha)^2$

$= (cos^2\beta - 2cos\alpha cos\beta + cos^2\alpha) + (sin^2\beta - 2sin\alpha sin\beta + sin^2\alpha)$

$= (cos^2\beta + sin^2\beta) + (cos^2\alpha + sin^2\alpha) - 2(cos\alpha cos\beta +$

$sin\alpha sin\beta)$

$= 2(1 - \cos(\beta - \alpha))$

Then, since $1 - cos2\theta = 2sin^2\theta$,

$\left|e^{i\beta}-e^{i\alpha}\right|^2 = 4sin^2[\frac{1}{2}(\beta-\alpha)]$

and $\left|e^{i\beta}-e^{i\alpha}\right| = 2sin[\frac{1}{2}(\beta-\alpha)]$,

since $0<\alpha<\beta<2\pi \Rightarrow sin[\frac{1}{2}(\beta-\alpha)]>0$ (and we require the +ve square root).

2nd Part

Hence $\left|e^{i\alpha}-e^{i\beta}\right|\left|e^{i\gamma}-e^{i\delta}\right|+\left|e^{i\beta}-e^{i\gamma}\right|\left|e^{i\alpha}-e^{i\delta}\right|$

$=\left|e^{i\beta}-e^{i\alpha}\right|\left|e^{i\delta}-e^{i\gamma}\right|+\left|e^{i\gamma}-e^{i\beta}\right|\left|e^{i\delta}-e^{i\alpha}\right|$

[The official sol'ns seem to overlook the need for the above step.]

$= 2sin\left[\frac{1}{2}(\beta-\alpha)\right]2sin\left[\frac{1}{2}(\delta-\gamma)\right]$

$+2sin[\frac{1}{2}(\gamma-\beta)]2sin[\frac{1}{2}(\delta-\alpha)]$ (*)

As $\cos(A-B)-\cos(A+B) = 2sinAsinB$,

$(*) = 2\left\{cos\frac{1}{2}(\beta-\alpha-\delta+\gamma) - cos\frac{1}{2}(\beta-\alpha+\delta-\gamma)\right\}$

$+2\{cos\frac{1}{2}(\gamma-\beta-\delta+\alpha) - cos\frac{1}{2}(\gamma-\beta+\delta-\alpha)\}$

$= 2\left\{cos\frac{1}{2}(\beta-\alpha-\delta+\gamma) - cos\frac{1}{2}(\beta-\alpha+\delta-\gamma)\right\}$

$+2\{cos\frac{1}{2}(\beta-\gamma+\delta-\alpha) - cos\frac{1}{2}(\beta-\gamma+\alpha-\delta)\}$

$= 2\left\{cos\frac{1}{2}(\beta-\alpha-\delta+\gamma) - cos\frac{1}{2}(\beta-\gamma+\alpha-\delta)\right\}$ (**)

Also, $\left|e^{i\alpha}-e^{i\gamma}\right|\left|e^{i\beta}-e^{i\delta}\right| = \left|e^{i\gamma}-e^{i\alpha}\right|\left|e^{i\delta}-e^{i\beta}\right|$

$= 2sin\left[\frac{1}{2}(\gamma-\alpha)\right]2sin\left[\frac{1}{2}(\delta-\beta)\right]$

$$2\left\{\cos\frac{1}{2}(\gamma-\alpha-\delta+\beta)-\cos\frac{1}{2}(\gamma-\alpha+\delta-\beta)\right\}=(**)$$

Hence $\left|e^{i\alpha}-e^{i\beta}\right|\left|e^{i\gamma}-e^{i\delta}\right|+\left|e^{i\beta}-e^{i\gamma}\right|\left|e^{i\alpha}-e^{i\delta}\right|$

$=\left|e^{i\alpha}-e^{i\gamma}\right|\left|e^{i\beta}-e^{i\delta}\right|$, as required

The complex numbers $e^{i\alpha}, e^{i\beta}, e^{i\gamma}$ & $e^{i\delta}$ are points on a circle of radius 1, centre the origin (anti-clockwise in that order).

Referring to the diagram below, the result says that $pr+qs=xy$;

ie the sum of the products of opposite sides equals the product of the diagonals

This can be extended to any cyclic quadrilateral (ie of any radius, and centre) by choosing a suitable scale.