STEP 2009, Paper 3, Q6 - Solution (2 pages; 7/6/18)

$\left|e^{i\beta} - e^{i\alpha}\right| = |cos\beta + isin\beta - (cos\alpha + isin\alpha)|$

So $\left|e^{i\beta} - e^{i\alpha}\right|^2 = (cos\beta - cos\alpha)^2 + (sin\beta - sin\alpha)^2$

$= (cos^2\beta - 2cos\alpha cos\beta + cos^2\alpha) + (sin^2\beta - 2sin\alpha sin\beta + sin^2\alpha)$

$= (cos^2\beta + sin^2\beta) + (cos^2\alpha + sin^2\alpha) - 2(cos\alpha cos\beta + sin\alpha sin\beta)$

$= 2(1 - \cos(\beta - \alpha))$

Then, since $1 - cos2\theta = 2sin^2\theta$,

$\left|e^{i\beta} - e^{i\alpha}\right|^2 = 4sin^2[\frac{1}{2}(\beta - \alpha)]$

and $\left|e^{i\beta} - e^{i\alpha}\right| = 2sin[\frac{1}{2}(\beta - \alpha)]$,

since $0 < \alpha < \beta < 2\pi \Rightarrow sin[\frac{1}{2}(\beta - \alpha)] > 0$ (and we require the +ve square root).

Hence $\left|e^{i\alpha} - e^{i\beta}\right|\left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right|\left|e^{i\alpha} - e^{i\delta}\right|$

$= \left|e^{i\beta} - e^{i\alpha}\right|\left|e^{i\delta} - e^{i\gamma}\right| + \left|e^{i\gamma} - e^{i\beta}\right|\left|e^{i\delta} - e^{i\alpha}\right|$

[The official sol'ns seem to overlook the need for the above step.]

$= 2sin\left[\frac{1}{2}(\beta - \alpha)\right]2sin\left[\frac{1}{2}(\delta - \gamma)\right] + 2sin[\frac{1}{2}(\gamma - \beta)]2sin[\frac{1}{2}(\delta - \alpha)]$ (*)

As $\cos(A - B) - \cos(A + B) = 2sinAsinB$,

(*) $= 2\left\{cos\frac{1}{2}(\beta - \alpha - \delta + \gamma) - cos\frac{1}{2}(\beta - \alpha + \delta - \gamma)\right\}$

$+2\{cos\frac{1}{2}(\gamma - \beta - \delta + \alpha) - cos\frac{1}{2}(\gamma - \beta + \delta - \alpha)\}$

$= 2\left\{cos\frac{1}{2}(\beta - \alpha - \delta + \gamma) - -cos\frac{1}{2}(\gamma - \beta + \delta - \alpha)\right\}$, (**)

since $cos\frac{1}{2}(\gamma - \beta - \delta + \alpha) = \cos\left[-\frac{1}{2}(\gamma - \beta - \delta + \alpha)\right]$

$= cos\frac{1}{2}(\beta - \alpha + \delta - \gamma)$

Also, $\left|e^{i\alpha} - e^{i\gamma}\right|\left|e^{i\beta} - e^{i\delta}\right| = \left|e^{i\gamma} - e^{i\alpha}\right|\left|e^{i\delta} - e^{i\beta}\right|$

$= 2sin\left[\frac{1}{2}(\gamma - \alpha)\right]2sin\left[\frac{1}{2}(\delta - \beta)\right]$

$2\left\{cos\frac{1}{2}(\gamma - \alpha - \delta + \beta) - cos\frac{1}{2}(\gamma - \alpha + \delta - \beta)\right\} = (**)$

Hence $\left|e^{i\alpha} - e^{i\beta}\right|\left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right|\left|e^{i\alpha} - e^{i\delta}\right|$

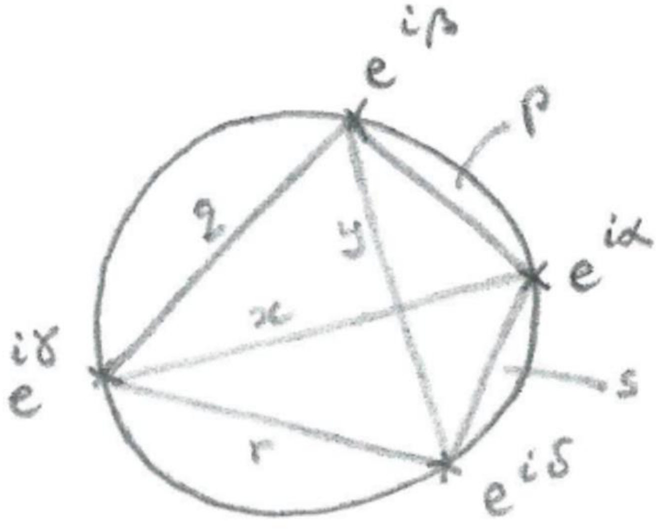
$= \left|e^{i\alpha} - e^{i\gamma}\right|\left|e^{i\beta} - e^{i\delta}\right|$, as required

The complex numbers $e^{i\alpha}, e^{i\beta}, e^{i\gamma}$ & $e^{i\delta}$ are points on a circle of radius 1, centre the origin (anti-clockwise in that order).

Referring to the diagram below, the result says that $pr + qs = xy$;

ie the sum of the products of opposite sides equals the product of the diagonals

This can be extended to any cyclic quadrilateral (ie of any radius, and centre) by choosing a suitable scale.



$e^{i\beta}$
p
q
y
$e^{i\alpha}$
x
$e^{i\gamma}$
s
r
$e^{i\delta}$