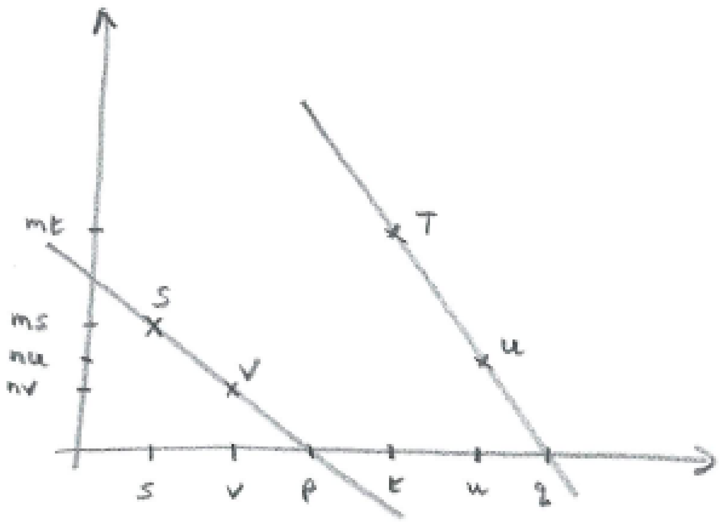
STEP 2009, Paper 3, Q1 - Solution (2 pages; 7/6/2018)

[The following diagram can be drawn 'without loss of generality' – even though we are told later on that S,T,U&V lie on a circle



It is possible to derive the equation of the line SV and then substitute in y=0 to find p. However, a short cut is to equate two expressions for the gradient:

$\frac{nv-ms}{v-s} = \frac{-nv}{p-v}$

so that p-v = $\frac{nv(v-s)}{ms-nv}$ and p = $\frac{v(nv-ns+ms-nv)}{ms-nv} = \frac{sv(m-n)}{ms-nv}$, as required

In exactly the same way, q = $\frac{tu(m-n)}{mt-nu}$

s and t are the two roots of the quadratic equation:

$x^2+(mx-c)^2 = r^2$

or $(1+m^2)x^2$ -2mcx + $(c^2$-$r^2)$ = 0

Hence s+t = - (-2mc)/(1+m^2) = 2mc/(1+m^2)

and st = $(c^2$-r^2)/ (1+m^2)

In exactly the same way, v+u = 2nc/(1+n^2)

and vu = (c^2-r^2)/ (1+n^2)

Then p+q = $\frac{sv(m-n)}{ms-nv} + \frac{tu(m-n)}{mt-nu}$

$= \frac{(m-n)}{(ms-nv)(mt-nu)}$ {mstv – nsuv +mstu – ntuv}

The expression in curly brackets =

mst(u+v) – nuv(s+t) = $\frac{m(c^2-r^2).2nc}{(1+m^2)(1+n^2)}$ - $\frac{n(c^2-r^2).2mc}{(1+n^2)(1+m^2)}$ = 0, as required