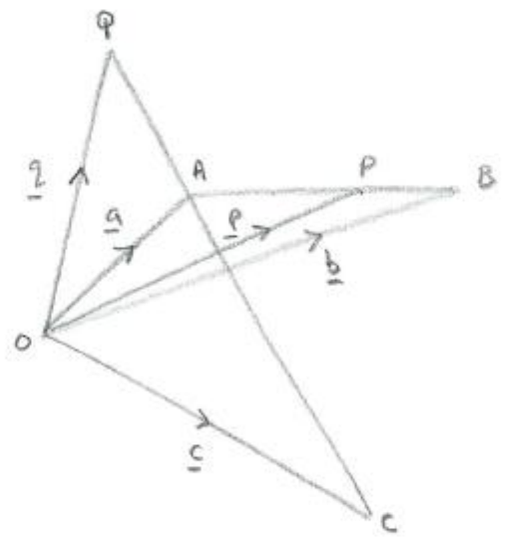
STEP 2009, Paper 2, Q8 – Solution (3 pages; 5/6/18)

[According to the ER, this question was not at all popular. Although a fairly complicated diagram emerges, the problem is solved by vector methods, rather than geometry.]

[A useful device where unknown values λ & μ are involved is to draw the diagram with specific values of λ & μ in mind - which may help to make the problem less abstract; and in this case ensures that the diagram satisfies the given constraints on λ & μ.]

[Note that $\underline{p} = \lambda\underline{a} + (1-\lambda)\underline{b}$ can be written as $\underline{b} + \lambda(\underline{a} - \underline{b})$, making it clear that, when $0 < \lambda < 1$, P must lie between A and B. Similarly, $\underline{q} = \underline{c} + \mu(\underline{a} - \underline{c})$, so that $\mu > 1$ means that Q must lie on the opposite side of A from C.]



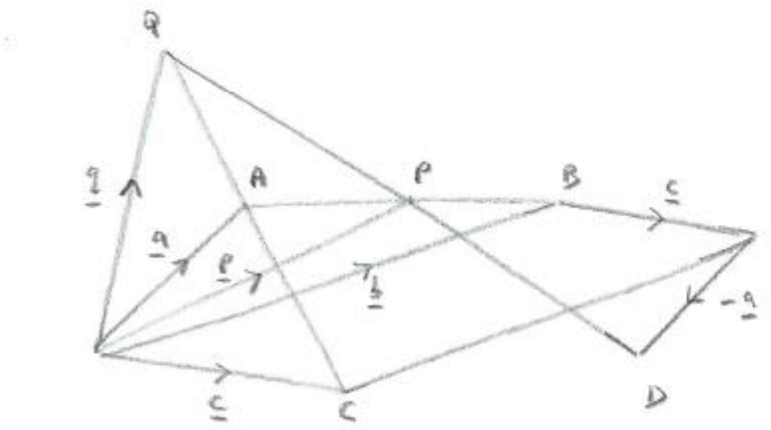
[CQ, BP etc are scalar lengths; in any case, the vector product is not in the STEP 2 syllabus]

$CQ \times BP = AB \times AC \Leftrightarrow \left|\underline{q} - \underline{c}\right| \left|\underline{p} - \underline{b}\right| = |\underline{a} - \underline{b}||\underline{a} - \underline{c}|$

$\Leftrightarrow |\mu(\underline{a} - \underline{c})||\lambda(\underline{a} - \underline{b})| = |\underline{a} - \underline{b}||\underline{a} - \underline{c}|$

so that $\mu\lambda = \pm 1$; then, as $0 < \lambda < 1$ & $\mu > 1$, $\mu = \frac{1}{\lambda}$

[Although a diagram isn't essential for the next part, we can't be sure of this in advance; and in fact it is useful for the last part. In order for the diagram to satisfy the constraint $CQ \times BP = AB \times AC$, we can (secretly) choose, for example, $\lambda = \frac{1}{2}$ & $\mu = 2$]



In order to show that D lies on QP extended, we use the standard vector device that

$\underline{d} = \underline{p} + k(\underline{p} - \underline{q})$ for some k

[In other situations this is often beneficial: although we are introducing an extra parameter k, the vector equation can often be turned into two equations, by equating components.]

$\underline{p} + k\left(\underline{p} - \underline{q}\right) = (1+k)\left(\lambda\underline{a} + (1-\lambda)\underline{b}\right) - k(\mu\underline{a} + (1-\mu)\underline{c}$

$= \underline{a}([1+k]\lambda - k\mu) + \underline{b}(1+k)(1-\lambda) - k(1-\mu)\underline{c}$

In order for this to equal $-\underline{a} + \underline{b} + \underline{c}$, let $1 + k = \frac{1}{1-\lambda}$, so that the coefficient of $\underline{b}$ is correct.

Then the coefficient of $\underline{a}$ is $\frac{\lambda}{1-\lambda} - \left(\frac{1}{1-\lambda} - 1\right)\left(\frac{1}{\lambda}\right) = \frac{\lambda}{1-\lambda} - \frac{\lambda}{1-\lambda}\left(\frac{1}{\lambda}\right)$

$= \frac{\lambda-1}{1-\lambda} = -1$, as required;

and the coefficient of $\underline{c}$ is $-\left(\frac{1}{1-\lambda} - 1\right)\left(1 - \frac{1}{\lambda}\right) = -\frac{\lambda}{1-\lambda}\left(\frac{\lambda-1}{\lambda}\right) = 1$,

as required.

From the 2nd diagram, ABDC would appear to be a parallelogram.

To prove this, we require $\underline{b} - \underline{a} = \underline{d} - \underline{c}$, and this follows from $\underline{d} = -\underline{a} + \underline{b} + \underline{c}$

[also $\underline{c} - \underline{a} = \underline{d} - \underline{b}$]