STEP 2009, Paper 2, Q5 – Solution (2 pages; 5/6/18)

[Q5 Note that the $\sqrt{}$ symbol denotes the positive root.]

$(\sqrt{x-1}+1)^2 = x-1+2\sqrt{x-1}+1 = x+2\sqrt{x-1}$

(i) [Note that the $\sqrt{}$ symbol denotes the positive root]

$\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$, since $\sqrt{x-1}+1>0$

Similarly, $(\sqrt{x-1}-1)^2 = x-1-2\sqrt{x-1}+1 = x-2\sqrt{x-1}$

Then $\sqrt{x-2\sqrt{x-1}} = \sqrt{x-1}-1$, provided $\sqrt{x-1}-1 \geq 0$ (A)

In the range $5 \leq x \leq 10$, $\sqrt{x-1}-1 \geq 1$, so that (A) is satisfied.

Then $\int_5^{10} \frac{\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}\ dx = \int_5^{10} \frac{\sqrt{x-1}+1+\sqrt{x-1}-1}{\sqrt{x-1}}\ dx$

$= \int_5^{10} 2\ dx = 2(10-5) = 10$

(ii)

(A) is satisfied when $x \geq 2$

For $x<2$, $\sqrt{x-2\sqrt{x-1}} = -(\sqrt{x-1}-1) = 1-\sqrt{x-1}$

So the area $= \int_{\frac{5}{4}}^{2} \frac{1-\sqrt{x-1}}{\sqrt{x-1}}\ dx + \int_2^{10} \frac{\sqrt{x-1}-1}{\sqrt{x-1}}\ dx$

$= \left[\frac{\sqrt{x-1}}{1/2} - x\right]_{5/4}^{2} + \left[x - \frac{\sqrt{x-1}}{1/2}\right]_{2}^{10}$

$= (2-2) - \left(1-\frac{5}{4}\right) + (10-6) - (2-2) = \frac{17}{4}$

[The official sol'ns claim that part of the area lies below the x −axis. However, because the square root represents the positive root, $y = \frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}$ is never negative.]

(iii) Once again, $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ for $5/4 \leq x \leq 10$

Then $\left(\sqrt{x+1}-1\right)^2 = x+1-2\sqrt{x+1}+1$, so that $\sqrt{x-2\sqrt{x+1}+2} = \sqrt{x+1}-1$ for $5/4 \leq x \leq 10$

(since $\sqrt{x+1}-1$ is positive in this range)

Hence $\int_{5/4}^{10} \frac{\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x+1}+2}}{\sqrt{x^2-1}}\ dx = \int_{5/4}^{10} \frac{\sqrt{x-1}+1+\sqrt{x+1}-1}{\sqrt{x^2-1}}\ dx$

$= \int_{5/4}^{10} \frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x-1}}\ dx = \left[\frac{\sqrt{x+1}}{1/2} + \frac{\sqrt{x-1}}{1/2}\right]_{5/4}^{10}$

$= \left(2\sqrt{11}+6\right) - (3+1) = 2\sqrt{11}+2$

[It's unusual for there to be no complication in the last part, so you could be forgiven for thinking that you had missed something.]