STEP 2009, Paper 1, Q2 - Solution (2 pages; 5/6/18)

$y^3 = x^3 + a^3 + b^3 \Rightarrow 3y^2 \frac{dy}{dx} = 3x^2$

so that $\frac{dy}{dx}$ at $(-a, b) = \frac{a^2}{b^2}$

and the eq'n of the tangent at this point is

$y - b = \frac{a^2}{b^2}(x - [-a]) \Rightarrow b^2 y - b^3 = a^2 x + a^3$

$\Rightarrow b^2 y - a^2 x = a^3 + b^3$, as required (1)

When the tangent meets the curve (for $a = 1, b = 2$),

$4y - x = 9$ & $y^3 = x^3 + 9$

so that $(4y)^3 = (x + 9)^3$ & $(4y)^3 = 64(x^3 + 9)$

$\Rightarrow (x + 9)^3 = 64(x^3 + 9)$

$\Rightarrow x^3(64 - 1) + x^2(-27) + x(-3)(81) + 9(64 - 81) = 0$

$\Rightarrow 7x^3 - 3x^2 - 27x - 17 = 0$, as required (2)

Putting $p = y$, $q = x, r = a = 1$ & $s = b = 2$, (2) provides us with values satisfying $p^3 = q^3 + r^3 + s^3$ (3)

We know that the tangent meets the curve when $x = -a = -1$, but as the values in (3) must be positive, we need to look for a positive root of (2).

We can write $7x^3 - 3x^2 - 27x - 17 = (x + 1)(7x^2 + kx - 17)$

Equating coefficients of x^2: $-3 = k + 7$,

so that $k = -10$

Then $7x^2 - 10x - 17 = (7x - 17)(x + 1)$

[looking for a & b such that $a+b=-10$ & $ab=(7)(-17)$

ie -17 & 7, and noting that, with $(7x+\alpha)(x+\beta)$, β has to be 1 in order to give one of $-17x$ & $7x$]

[The repeated root of -1 could in fact have been deduced from the fact that we have a tangent to the curve (ie rather than just a line intersecting the curve).]

Thus, $\frac{17}{7}$ is a positive root and, when $x=\frac{17}{7}$, $4y-\frac{17}{7}=1+8$,

from (1), so that $y=\frac{1}{4}\left(9+\frac{17}{7}\right)=\frac{80}{28}=\frac{20}{7}$

Then, from (3), $\left(\frac{20}{7}\right)^3=\left(\frac{17}{7}\right)^3+1^3+2^3$

and hence $20^3=17^3+7^3+14^3$