STEP 2008, Paper 2, Q7 – Solution (3 pages; 2/6/18)

[As (ii) is likely to involve a similar substitution to (i), it would help if we could identify the features of the substitution in (i) that make it work.]

(i) Let $y = u(1+x^2)^{\frac{1}{2}}$, so that $\frac{dy}{dx} = \frac{du}{dx}(1+x^2)^{\frac{1}{2}} + \frac{2u}{2}(1+x^2)^{-\frac{1}{2}}x$

Then $\frac{1}{y}\frac{dy}{dx} = xy + \frac{x}{1+x^2} \Rightarrow$

$\frac{du}{dx}(1+x^2)^{\frac{1}{2}} + u(1+x^2)^{-\frac{1}{2}}x = xu^2(1+x^2) + \frac{x}{1+x^2}\,u(1+x^2)^{\frac{1}{2}}$

$\Rightarrow \frac{du}{dx}(1+x^2) + ux = xu^2(1+x^2)^{\frac{3}{2}} + xu$

$\Rightarrow \int \frac{1}{u^2}\,du = \int x(1+x^2)^{\frac{1}{2}}\,dx$

$\Rightarrow -\frac{1}{u} = \frac{1}{2}\frac{(1+x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c$

$x = 0, y = 1 \Rightarrow x = 0, u = 1$

so that $-1 = \frac{1}{3} + c$, and hence $c = -\frac{4}{3}$

and $\frac{1}{u} = \frac{1}{3}\left(4 - (1+x^2)^{\frac{3}{2}}\right)$,

so that $y = \frac{3(1+x^2)^{\frac{1}{2}}}{4-(1+x^2)^{\frac{3}{2}}}$

(ii) The $1+x^2$ in the substitution in (i) was needed to combine with the $1+x^2$ appearing on the right-hand side of the differential equation. So this strongly suggests that a substitution of the form $y = u(1+x^3)^k$ is required here. $k = \frac{1}{3}$ has the advantage that it cancels with the 3 from the derivative of x^3, in the same way that the power of $\frac{1}{2}$ cancelled with the 2 from the derivative of x^2 in (i).

A safe method is to make a substitution of $y = u(1+x^3)^k$ and see what value of k is needed (bearing in mind that the substitution in (i) worked because of the ux term appearing on both sides).

Let $y = u(1+x^3)^k$,

so that $\frac{dy}{dx} = \frac{du}{dx}(1+x^3)^k + ku(1+x^3)^{k-1}(3x^2)$

Then $\frac{1}{y}\frac{dy}{dx} = x^2 y + \frac{x^2}{1+x^3} \Rightarrow$

$\frac{du}{dx}(1+x^3)^k + ku(1+x^3)^{k-1}(3x^2)$

$= x^2u^2(1+x^3)^{2k} + \frac{x^2}{1+x^3}\, u(1+x^3)^k$

In order for the 2nd term on the LHS to cancel with the 2nd term on the RHS, we must have $k = \frac{1}{3}$, as expected.

Then $\frac{du}{dx}(1+x^3)^{\frac{1}{3}} = x^2u^2(1+x^3)^{\frac{2}{3}}$

and hence $\frac{du}{dx} = x^2u^2(1+x^3)^{\frac{1}{3}}$

$\Rightarrow \int \frac{1}{u^2}\,du = \frac{1}{3}\int 3x^2(1+x^3)^{\frac{1}{3}}\,dx$

$\Rightarrow -\frac{1}{u} = \frac{1}{3}\frac{(1+x^3)^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} + c$

$x = 0, y = 1, u = 1 \Rightarrow -1 = \frac{1}{4} + c \Rightarrow c = -\frac{5}{4}$

So $\frac{1}{u} = \frac{1}{4}(5 - (1+x^3)^{\frac{4}{3}})$

$\Rightarrow y = \frac{4(1+x^3)^{\frac{1}{3}}}{5-(1+x^3)^{\frac{4}{3}}}$

(iii) $y = \frac{(n+1)(1+x^n)^{\frac{1}{n}}}{(n+2)-(1+x^n)^{\frac{n+1}{n}}}$

[It would seem that this is all that is needed.]