STEP 2008, Paper 2, Q6 – Solution (2 pages; 2/6/18)

[Note that, unlike a function of the form

$cos\left(2x+\frac{\pi}{3}\right)+sin(2x-\frac{\pi}{4})$, or even

$acos\left(2x+\frac{\pi}{3}\right)+bsin(2x-\frac{\pi}{4})$, we can't write this as

$Asin(cx+d)$.]

(i) The period T_1 of $cos\left(2x+\frac{\pi}{3}\right)$ satisfies $2T_1=2\pi$ [as
$\cos\left(2[0]+\frac{\pi}{3}\right)=\cos\left(2\pi+\frac{\pi}{3}\right)$]; ie $T_1=\pi$

Similarly for $sin(\frac{3x}{2}-\frac{\pi}{4}), \frac{3T_2}{2}=2\pi$, so that $T_2=\frac{4\pi}{3}$

The period of $f(x)$ is the LCM of these two periods; ie 4π.

(ii) [The official sol'ns point out that $f(x)$ can be written in the form $2cosCcosD$, where $C=\frac{1}{2}(A+B), D=\frac{1}{2}(A-B)$ and

$A=2x+\frac{\pi}{3}\ , B=\frac{3x}{2}-\frac{\pi}{4}$]

$f(x)=0\Rightarrow \sin\left(\frac{\pi}{2}-\left[2x+\frac{\pi}{3}\right]\right)=-sin\left(\frac{3x}{2}-\frac{\pi}{4}\right)=\sin(\frac{\pi}{4}-\frac{3x}{2})$

[Alternatively, $sin\left(\frac{3x}{2}-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{2}-\left[\frac{3x}{2}-\frac{\pi}{4}\right]\right)$. Writing everything in terms of cos has the advantage that the general solution can be written down quickly;

ie $cosA=cosB\Rightarrow A=\pm B+2k\pi$

However, $-cosA$ is less easy to manipulate than $-sinA$]

$\Rightarrow \frac{\pi}{2}-\left[2x+\frac{\pi}{3}\right]=\frac{\pi}{4}-\frac{3x}{2}+2\pi k$ or $\pi-(\frac{\pi}{4}-\frac{3x}{2})+2\pi\lambda$

So either $\frac{x}{2}=\frac{\pi}{6}-\frac{\pi}{4}-2\pi k$ or $\frac{7x}{2}=\frac{\pi}{6}-\pi+\frac{\pi}{4}-2\pi\lambda$

ie $6x=2\pi-3\pi-24\pi k$ or $42x=2\pi-12\pi+3\pi-24\pi\lambda$

ie $6x = -\pi - 24\pi k$ or $42x = -7\pi - 24\pi\lambda$

ie $x = -\frac{\pi}{6} - 4\pi k$ or $x = -\frac{\pi}{6} - \frac{4\pi\lambda}{7}$

The sol'ns in the range $-\pi \leq x \leq \pi$ are:

$k = 0 \Rightarrow -\frac{\pi}{6}$

$\lambda = 0 \Rightarrow -\frac{\pi}{6}$

$\lambda = 1 \Rightarrow -\frac{31\pi}{42}$

$\lambda = -1 \Rightarrow \frac{17\pi}{42}$

$\lambda = -2 \Rightarrow \frac{41\pi}{42}$

The repeated root of $-\frac{\pi}{6}$ is where the curve touches the x axis.

(iii) $f(x) = 2 \Rightarrow cos\left(2x + \frac{\pi}{3}\right) = sin\left(\frac{3x}{2} - \frac{\pi}{4}\right) = 1$

$\Rightarrow 2x + \frac{\pi}{3} = 2k\pi \;\; and \;\; \frac{3x}{2} - \frac{\pi}{4} = \frac{\pi}{2} + 2\lambda\pi, \;\; 0 \leq x \leq 2\pi$

$\Rightarrow x = \pi\left(k - \frac{1}{6}\right) \;\; and \;\; x = \frac{2}{3}\left(\frac{3\pi}{4} + 2\lambda x\right) = \pi\left(\frac{1}{2} + \frac{4\lambda}{3}\right)$

$\Rightarrow \{x = \frac{5\pi}{6} \; (k = 1) \; or \; \frac{11\pi}{6} \; (k = 2)\}$

and $\{x = \frac{\pi}{2} \; (\lambda = 0) \; or \; \frac{11\pi}{6} \; (\lambda = 1)\}$

Therefore $x = \frac{11\pi}{6}$

[Unusually, the result in (i) doesn't seem to be needed for (ii) or (iii).]