STEP 2008, Paper 2, Q5 – Solution (2 pages; 2/6/18)

Hint: Look for $f'(x)$ in the numerator of the integrals.

Solution

$1 + sin^2x = 1 + \frac{1}{2}(1 - cos2x) = \frac{3}{2} - \frac{1}{2}cos2x$

So $I_1 = -\int_0^{\frac{\pi}{2}} \frac{-2sin2x}{3-cos2x}\ dx = -[-ln(3-u)]_1^{-1}$

[making the substitution $u = cos2x$]

$= (ln4 - ln2) = 2ln2 - ln2 = ln2$

$I_2 = \int_0^{\frac{\pi}{2}} \frac{sinx}{2-cos^2x} dx$; let $u = cosx$

so that $I_2 = -\int_1^0 \frac{1}{2-u^2} du = \frac{1}{2\sqrt{2}} \int_0^1 \frac{1}{\sqrt{2}-u} + \frac{1}{\sqrt{2}+u}\ du$

$= \frac{1}{2\sqrt{2}} [-\ln(\sqrt{2}-u) + \ln(\sqrt{2}+u)]_0^1$

$= \frac{1}{2\sqrt{2}} \{(-\ln(\sqrt{2}-1) + \ln(\sqrt{2}+1)) - 0\}$

$= \frac{1}{2\sqrt{2}} ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{1}{2\sqrt{2}} \ln(\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{2-1} = \frac{1}{2\sqrt{2}} \ln(2 + 1 + 2\sqrt{2})$

$= \frac{1}{2\sqrt{2}} \ln(3 + 2\sqrt{2})$

[This equals the expression $\frac{1}{\sqrt{2}} \ln(1+\sqrt{2})$ given in the official solutions, as $(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$, so that $\sqrt{3+2\sqrt{2}} = 1 + \sqrt{2}$]

$(1+\sqrt{2})^5 = 1 + 5\sqrt{2} + 10(2) + 10(2)\sqrt{2} + 5(4) + 4\sqrt{2}$

$= 41 + 29\sqrt{2} < 41 + 29(2) = 99$, as required

$(1.4)^2 = 1 + 0.16 + 0.8 = 1.96$

Hence $1.4 = \sqrt{1.96} < \sqrt{2}$, as required

$2^{\sqrt{2}} > 2^{1.4} = 2^{\frac{7}{5}} = (2^7)^{\frac{1}{5}} = \sqrt[5]{128} > \sqrt[5]{99} > 1 + \sqrt{2}$, as required

(from the 1st result)

$I_1 = \frac{1}{\sqrt{2}} ln\left(2^{\sqrt{2}}\right)$; $I_2 = \frac{1}{\sqrt{2}} \ln(3 + 2\sqrt{2})^{\frac{1}{2}}$

To find which is the larger of $2^{\sqrt{2}}$ & $(3 + 2\sqrt{2})^{\frac{1}{2}}$:

From the previous result, $2^{\sqrt{2}} > 1 + \sqrt{2}$,

so that $\left(2^{\sqrt{2}}\right)^2 > \left(1 + \sqrt{2}\right)^2 = 1 + 2 + 2\sqrt{2} = 3 + 2\sqrt{2}$

and hence $2^{\sqrt{2}} > (3 + 2\sqrt{2})^{\frac{1}{2}}$

Therefore $I_1 > I_2$, as $y = lnx$ is an increasing function.