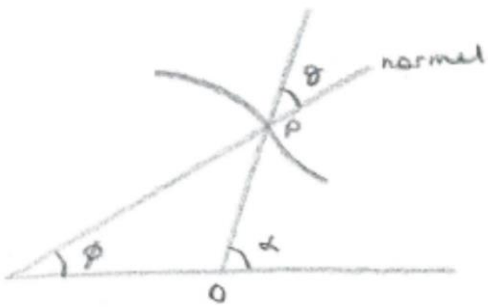
STEP 2008, Paper 2, Q4 – Solution (2 pages; 2/6/18)

Differentiating $x^2 + y^2 + 2axy = 1$ implicitly:

$2x + 2y\frac{dy}{dx} + 2ay + 2ax\frac{dy}{dx} = 0$

$\Rightarrow x + ay + \frac{dy}{dx}(ax + y) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-(x+ay)}{ax+y}$, as required (provided that $ax + y \neq 0$)



Let $tan\phi$ be the gradient of the normal, and $tan\alpha$ the gradient of OP. Then, if $\alpha > \phi$, $\theta = \alpha - \phi$ [as in the diagram] and if $\phi > \alpha$, $\theta = \phi - \alpha$

So $\theta = |\phi - \alpha|$ and $\tan\theta = tan|\phi - \alpha| = |\tan(\phi - \alpha)|$

$= \left|\frac{tan\phi - tan\alpha}{1 + tan\phi tan\alpha}\right| = \left|\frac{\frac{ax+y}{x+ay} - \frac{y}{x}}{1 + \left(\frac{ax+y}{x+ay}\right)\left(\frac{y}{x}\right)}\right|$

$= \left|\frac{x(ax+y) - y(x+ay)}{(x+ay)x + (ax+y)y}\right|$

$= \left|\frac{ax^2 - ay^2}{x^2 + y^2 + 2axy}\right|$

$= a|y^2 - x^2|$, as $x^2 + y^2 + 2axy = 1$ (given) and $a > 0$

as required

(i) If $y > x$, then $tan\theta = a(y^2 - x^2)$

Differentiating $\Rightarrow sec^2\theta\,\frac{d\theta}{dx} = 2ay\frac{dy}{dx} - 2ax$ (1)

Given that $\frac{d\theta}{dx} = 0$ & $\frac{dy}{dx} = -\frac{x+ay}{ax+y}$,

(1) then $\Rightarrow y\left(-\frac{x+ay}{ax+y}\right) - x = 0$, since $a \neq 0$

$\Rightarrow y(x + ay) + x(ax + y) = 0$

$\Rightarrow a(x^2 + y^2) + 2xy = 0$, as required

If $y < x$, then $\sec^2\theta \frac{d\theta}{dx} = -(2ay\frac{dy}{dx} - 2ax)$,

but $\frac{d\theta}{dx} = 0$ produces the same result.

[For (ii), it isn't immediately obvious how to proceed. It might be necessary, for example, to differentiate again. But if in doubt just use the immediately preceding result.]

(ii) $(1 + a)(x^2 + y^2 + 2xy) = a(x^2 + y^2) + 2xy + x^2 + y^2 + 2axy$

$= 0 + 1 = 1$, from (i) & $x^2 + y^2 + 2axy = 1$ (given)

(iii) As $tan\theta = a|y^2 - x^2|$, $tan\theta = \frac{a}{\sqrt{1-a^2}} \Leftrightarrow (y^2 - x^2)^2 = \frac{1}{1-a^2}$ (2)

LHS of (2) $= (x + y)^2(y - x)^2 = (x^2 + y^2 + 2xy)(y^2 + x^2 - 2xy)$

$= \frac{1}{1+a}(y^2 + x^2 - 2xy)$, from (ii) (3)

[Consider the other equations not yet used:]

Consider

$x^2 + y^2 + 2axy = 1$ (given) (4)

and $a(x^2 + y^2) + 2xy = 0$ (from (i)) (5)

Write $u = x^2 + y^2$ & $v = 2xy$.

Then (4) & (5) become $u + av = 1$ (6) & $au + v = 0$ (7)

[We need to find an expression for $y^2 + x^2 - 2xy = u - v$]

(6) & (7) $\Rightarrow u + a(-au) = 1 \Rightarrow u = \frac{1}{1-a^2}$ & $v = -\frac{a}{1-a^2}$

Hence $y^2 + x^2 - 2xy = u - v = \frac{1}{1-a^2}(1 + a)$

Then, from (3), the LHS of (2) becomes $\frac{1}{1-a^2}$, as required.