**STEP 2008, Paper 2, Q3 – Solution** (2 pages; 2/6/18)

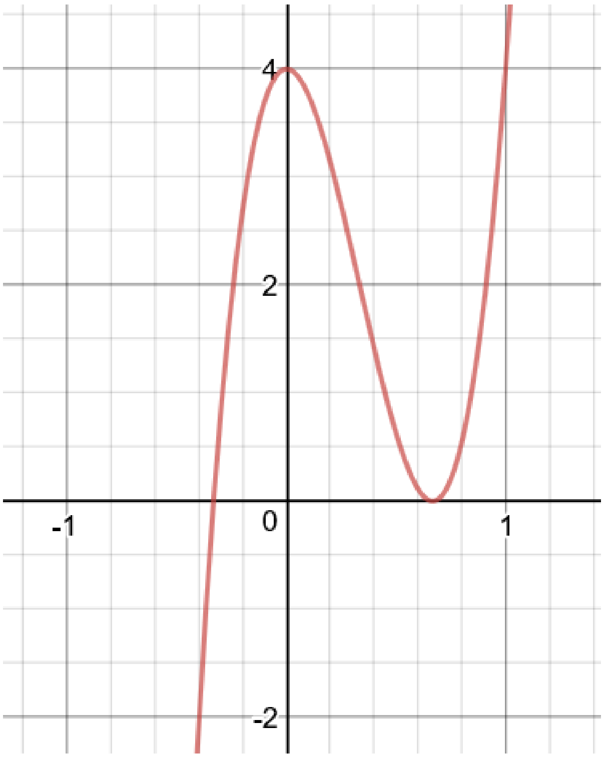
(i) $y = 27x^3 - 27x^2 + 4$

$\frac{dy}{dx} = 81x^2 - 54x = 27x(3x - 2)$

$\frac{dy}{dx} = 0 \Rightarrow x = 0 \; or \; x = \frac{2}{3}$

$x = \frac{2}{3} \Rightarrow y = 8 - 12 + 4 = 0$

So turning points are $(0,4)$ & $(\frac{2}{3}, 0)$




From the graph, when $x \geq 0$, $27x^3 - 27x^2 + 4 \geq 0$

$\Leftrightarrow 27x^2(x - 1) \geq -4$

$\Leftrightarrow x^2(1 - x) \leq \frac{4}{27}$, as required.

Suppose that all of the numbers are $> \frac{4}{27}$ (*)

Then $bc(1-a) > \frac{4}{27}$ and $a^2(1-a) \leq \frac{4}{27}$ (from the previous result), so that $bc(1-a) > a^2(1-a)$, and hence $bc > a^2$

Similarly, $ca > b^2$ & $ab > c^2$

Multiplying these 3 inequalities together, $(bc)(ca)(ab) > a^2b^2c^2$,

which gives the contradiction $a^2b^2c^2 > a^2b^2c^2$

Hence (*) is not true, and at least one of the numbers is $\leq \frac{4}{27}$, as required.

(ii) Consider $x(1-x) \leq \frac{1}{4}$

ie $-4x^2 + 4x \leq 1$ or $4x^2 - 4x + 1 \geq 0$

or $(2x-1)^2 \geq 0$

Thus $x(1-x) \leq \frac{1}{4}$ is true for all $x$ (**)

Suppose then that $p(1-q)$ & $q(1-p)$ are both $> \frac{1}{4}$

Then $p(1-q) > q(1-q)$, by (**), so that $p > q$

Also $q(1-p) > p(1-p)$, by (**), so that $q > p$, which gives a contradiction, and hence at least one of $p(1-q)$ & $q(1-p)$

is $\leq \frac{1}{4}$