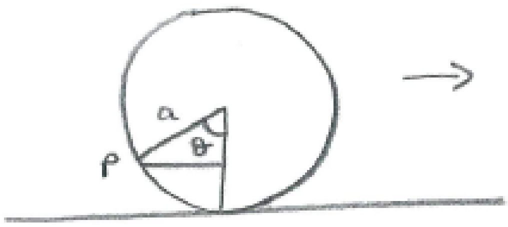
STEP 2008, Paper 1, Q9 - Solution (2 pages; 1/6/2018)

This question really requires past experience of rolling wheels.



[Suppose that the hoop is rotating in a clockwise sense. As P leaves the bottom of the hoop, it moves round to the left, and has speed a$\dot{\theta}$, along the circumference of the hoop, relative to the centre of the hoop. The ground that the hoop covers is equal to the distance that P moves along the circumference, and hence the centre of the hoop is also moving with speed a$\dot{\theta}$ (but to the right). The motion of P relative to the ground has two components: its motion relative to the centre of the hoop, and the motion of the centre of the hoop relative to the ground. When P is at the bottom of the hoop, its speed relative to the ground is therefore - a$\dot{\theta}$ + a$\dot{\theta} = 0$ (if motion to the right is considered to be positive); ie P is stationary! At the top of the hoop, P has speed a$\dot{\theta}$ + a$\dot{\theta}$ = 2a$\dot{\theta}$.]

Position of P = (- asinθ + aθ , -acosθ + a) = (a(θ-sinθ), a(1-cosθ))

[The position of P relative to the centre is (-asinθ , -acosθ); the position of the centre relative to the starting point of P is (aθ , a)]

['Speed' has to mean the magnitude of the velocity of P]

Differentiating wrt t, velocity of P = (a($\dot{\theta}$-cosθ. $\dot{\theta}$) , asinθ . $\dot{\theta}$)

$Speed^2 = a^2\dot{\theta}^2(1 - cos\theta)^2 + a^2sin^2\theta \, . \, \dot{\theta}^2$

$= a^2\dot{\theta}^2$ {1 -2cosθ +$cos^2\theta + sin^2\theta$}

$= 2a^2\dot{\theta}^2$(1- cosθ)

and Speed $= \text{a}\dot{\theta}\sqrt{2(1 - \cos\theta)}$ (A)

[As a refinement, $1 - \cos\theta$ can be shown to equal $2sin^2(\frac{\theta}{2})$, so that the Speed $= 2\ \text{a}\dot{\theta}|\sin(\frac{\theta}{2})|$ - noting that only the positive square root is wanted (as we are working out the magnitude of a vector).]

As gravity is the only force here that does any work, conservation of energy can be applied. Further, since any gain in the PE of one of the particles is matched by an equal loss of PE by the other particle, it follows that the total KE of the particles is constant.

Hence, the total KE equals the KE when P is at the bottom of the hoop and Q is at the top.

In this position, P has speed 0 and Q has speed 2a$\dot{\theta}$ (both from (A)),

so that the total KE = ½ m$(2a\dot{\theta})^2$ = 2m $a^2\dot{\theta}^2$ (B)

The ground that the hoop covers is equal to the distance that P (or Q) moves along the circumference (relative to the centre), and hence the hoop has speed $a\dot{\theta}$.

As the total KE is constant, $\dot{\theta}$ is constant (from (B)), and so the hoop rolls with constant speed.