STEP 2008, Paper 1, Q13 - Solution (5 pages;1/6/2018)

The examiner's report stresses the need for explanation with probability questions.

Possible approaches were:

(a) counting; each prob. is $\frac{X}{5!}$

(b) conditional probability (eg given $H_1$is in the 1st position, P(W_1 sits in position 2) $= \frac{1}{5}$)

(c) classification of the different seating patterns (as in the official sol'ns, for (ii))

Approach (c) is only applicable to this type of question, and is non-standard (ie likely to be unsuccessful). The examiner's report rates approach (b) as more challenging than (a), in the case of (ii).

In (iii), there is an obvious hint to use a 1- ... argument, The direct method is also possible, but turns out to be a bit fiddly (if done by the counting method). As the examiner's report points out, you could use one method as a check on the other (if you weren't under hideous time pressure).

The big risk of course with probability questions involving counting is of neglecting to multiply by the appropriate number, to allow for any symmetry involved.

This appears in at least two ways in this question: (a) multiplying by 3, having counted the number of ways of H_1W_1 being the only pair to be together, and (b) multiplying by 2, to allow for the order W_1H_1.

The circular aspect of the seating arrangement can be dealt with as follows:

(1) start with eg H_1 in the 1st position ("without loss of generality" - WLOG)

(2) when considering cases where H_1&W_1 are next to each other, the issue of W_1 being in the last position (and therefore next to H_1) can be avoided, by requiring W_1 to be in the 2nd position, and then multiplying by 2 to allow for the order W_1H_1

Alternative solution to the official one:

(i) Treat H_1W_1 etc as a single unit, and assume for the moment that all pairs are in this order.

Placing H_1W_1 in the 1st position WLOG, there are then 2 ways of adding the other pairs (either $(H_2W_2)(H_3W_3)$ or $(H_3W_3)(H_2W_2)$). As each of the 3 pairs can be the other way round, the number of ways is 2 x 2^3=16

Prob $= \frac{16}{5!} = \frac{2}{15}$

(ii) Consider eg $H_1W_1H_3H_2W_2W_3$, where H_3&W_3 are split up (WLOG H_1W_1 can be in the 1st two positions). We will therefore need to multiply by 3, to cover the possibilities of H_1&W_1 or H_2&W_2 being split up instead.

In the above arrangement, H_1&W_1 could be swapped, H_2&W_2 could be swapped, and H_3&W_3 could be swapped.

Hence, the total number of arrangements is 3x2x2x2 = 24.

Prob $= \frac{24}{5!} = \frac{1}{5}$

(iii) **Method A**: Find Prob(exactly 1 husband & wife pair sit together), by counting

Consider eg $H_1W_1W_2H_3H_2W_3$ & multiply by 3, to allow for H_2W_2 ... or H_3W_3... instead, and then by 2, to allow for W_1H_1 instead.

H_1 can be placed in the 1st position WLOG

After W_1 has been placed in the 2nd position, anyone else can be placed in position 3 (ie multiply by 4).

Then there are 2 possibilities for position 4 (H_3&W_3 in the above arrangement); ie multiply by 2.

This then fixes the last 2 positions.

Hence the total number of arrangements = 3x2x4x2=48

Required prob. $= 1 - \frac{16+24+48}{5!} = 1 - \frac{11}{15} = \frac{4}{15}$

Method B: Find Prob(exactly 1 husband & wife pair sit together), by conditional probability

Consider eg $H_1W_1W_2H_3H_2W_3$ & multiply by 3, to allow for H_2W_2 ... or H_3W_3... instead, and then by 2, to allow for W_1H_1 instead.

(**Note**: The 'conditional probability' approach thus involves some counting as well, since we are multiplying by 3x2.)

Having placed H_1 in the 1st position (WLOG), there is a $\frac{1}{5}$ probability of getting W_1 in the 2nd position.

Then there is a probability of 1 of getting a suitable person in the 3rd position.

Then there is a probability of $\frac{2}{3}$ of getting a suitable person in the 4th position ($H_3 or W_3$ in the above arrangement).

Then there is a probability of $\frac{1}{2}$ of getting a suitable person in the 5th position (in the above arrangement it has to be H_2, rather than W_3).

Hence, required probability $= 3 \times 2 \times \frac{1}{5} \times \frac{2}{3} \times \frac{1}{2} = \frac{2}{5} = \frac{48}{5!}$, as in method A.

Method C: direct , by counting

WLOG let H_1 be in the 1st position.

We need to exclude any cases where W_1 appears at the end (this is considered in (c)).

Possible patterns are:

(a) $H_1H_2W_1(H_3W_2W_3)$ - where W_1 is required in the 3rd position, and H_3&W_3 mustn't be together; giving 4 possibilities for the 2nd position $[H_2/H_3/W_2/W_3]$ and 2 possibilities for the last 3 positions; hence there are 4x2=8 total possibilities for (a).

(b) $H_1H_2H_3(W_1W_2W_3)$ - where W_1 is not in the 3rd position. Once again, there are 4 possibilities for the 2nd position $[H_2/H_3/W_2/W_3]$. Once the 2nd position has been filled, 2 of $H_2/H_3/W_2/W_3$ can then go in position 3.

We then have an arrangement of the form 123... or 132..., so that the only remaining constraint is that position 4 is not filled by the husband or wife of position 3. Thus there are 2x2 ways of filling the last 3 positions.

Hence there are 4x2x2x2=32 total possibilities for (b).

(c) We now have to consider the number of arrangements ending with W_1;

eg $H_1H_2H_3W_2W_3W_1$

As in (b), there are 4 possibilities for the 2nd position $[H_2/H_3/W_2/W_3]$ and then, once the 2nd position has been filled, 2 of $H_2/H_3/W_2/W_3$ can go in position 3. After that, there is no flexibility for the remaining positions.

Hence there are 4x2 = 8total possibilities for (c).

Thus there are $8 + 32 - 8 = 32$ possible arrangements, and hence the required probability $= \frac{32}{5!} = \frac{4}{15}$.

Note the use of the 'case by case' approach here, to simplify matters.