STEP 2007, Paper 3, Q1 – Solution (2 pages; 31/5/18)

$\tan(\theta + \phi) = \frac{tan\theta + tan\phi}{1 - tan\theta tan\phi}$

Let $\theta = \theta_1 + \theta_2$, $\phi = \theta_3 + \theta_4$

Then $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{\left(\frac{t_1 + t_2}{1 - t_1 t_2}\right) + \left(\frac{t_3 + t_4}{1 - t_3 t_4}\right)}{1 - \left(\frac{t_1 + t_2}{1 - t_1 t_2}\right)\left(\frac{t_3 + t_4}{1 - t_3 t_4}\right)}$

$= \frac{(t_1 + t_2)(1 - t_3 t_4) + (t_3 + t_4)(1 - t_1 t_2)}{(1 - t_1 t_2)(1 - t_3 t_4) - (t_1 + t_2)(t_3 + t_4)}$

$= \frac{(t_1 + t_2 + t_3 + t_4) - (t_1 t_3 t_4 + t_2 t_3 t_4 + t_1 t_2 t_3 + t_1 t_2 t_4)}{1 + t_1 t_2 t_3 t_4 - (t_1 t_2 + t_3 t_4 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4)}$

Then, as $t_1 + t_2 + t_3 + t_4 = -\frac{b}{a}$,

$t_1 t_2 + t_3 t_4 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 = \frac{c}{a}$,

$t_1 t_3 t_4 + t_2 t_3 t_4 + t_1 t_2 t_3 + t_1 t_2 t_4 = -\frac{d}{a}$ & $t_1 t_2 t_3 t_4 = \frac{e}{a}$,

$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{-\frac{b}{a} - \left(-\frac{d}{a}\right)}{1 + \frac{e}{a} - \frac{c}{a}} = \frac{d - b}{a + e - c}$ (1)

$pcos2\theta + \cos(\theta - \alpha) + p = 0$ (2)

$\Rightarrow pcos^2\theta - psin^2\theta + cos\theta cos\alpha + sin\theta sin\alpha = 0$

$\Rightarrow 2pcos^2\theta + cos\theta cos\alpha + sin\theta sin\alpha = 0$

$\Rightarrow 2pcos\theta + cos\alpha + tan\theta sin\alpha = 0$

(assuming that $cos\theta \neq 0$, so that $tan\theta = \frac{sin\theta}{cos\theta}$ is defined, as allowed by the instruction at the start of the question)

$\Rightarrow 2pcos\theta = -tan\theta sin\alpha - cos\alpha$

Let $t = tan\theta$,

so that $\frac{4p^2}{1+t^2} = (tsin\alpha + cos\alpha)^2$

$\Rightarrow 4p^2 = (1 + t^2)(t^2sin^2\alpha + 2tsin\alpha cos\alpha + cos^2\alpha)$

$\Rightarrow sin^2\alpha . t^4 + \sin(2\alpha)\, t^3 + (sin^2\alpha + cos^2\alpha)t^2 + \sin(2\alpha)\, t$

$+cos^2\alpha - 4p^2 = 0$

$\Rightarrow sin^2\alpha . t^4 + \sin(2\alpha)\, t^3 + t^2 + \sin(2\alpha)\, t + cos^2\alpha - 4p^2 = 0$ (3)

Thus, as θ_i satisfy (2), tanθ_i satisfy the polynomial (3).

Then, from (1), $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{\sin(2\alpha) - \sin(2\alpha)}{sin^2\alpha + (cos^2\alpha - 4p^2) - 1}$

$= \frac{0}{-4p^2} = 0$, provided that $p \neq 0$ (and this assumption is allowed by the instruction at the start of the question).

Hence $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.