STEP 2007, Paper 2, Q4 – Solution (6 pages; 29/3/25)

4 Given that $\cos A$, $\cos B$ and β are non-zero, show that the equation

$$\alpha\sin(A-B)+\beta\cos(A+B)=\gamma\sin(A+B)$$

reduces to the form

$$(\tan A-m)(\tan B-n)=0\,,$$

where m and n are independent of A and B, if and only if $\alpha^2=\beta^2+\gamma^2$.

Determine all values of x, in the range $0\leqslant x<2\pi$, for which:

(i) $2\sin(x-\frac{1}{4}\pi)+\sqrt{3}\cos(x+\frac{1}{4}\pi)=\sin(x+\frac{1}{4}\pi)\,;$

(ii) $2\sin(x-\frac{1}{6}\pi)+\sqrt{3}\cos(x+\frac{1}{6}\pi)=\sin(x+\frac{1}{6}\pi)\,;$

(iii) $2\sin(x+\frac{1}{3}\pi)+\sqrt{3}\cos(3x)=\sin(3x)\,.$

[The Examiner's Report makes no reference to the large amount of work needed to consider the cases where $cosA = 0$ or

$cosB = 0$. In the actual exam, it might be best to sacrifice any marks available for this.]

The equation can be written as

$\alpha sinAcosB - \alpha cosAsinB + \beta cosAcosB - \beta sinAsinB$

$-\gamma sinAcosB - \gamma cosAsinB = 0$

dividing by $cosAcosB$ (as $cosA$ & $cosB$ are non-zero)

$\Leftrightarrow \alpha tanA - \alpha tanB + \beta - \beta tanAtanB - \gamma tanA - \gamma tanB = 0$

$\Leftrightarrow tanAtanB - \frac{(\alpha-\gamma)tanA}{\beta} + \frac{(\alpha+\gamma)tanB}{\beta} - 1 = 0$ (as $\beta \neq 0$)

which, with $m = \frac{-(\alpha+\gamma)}{\beta}$ & $n = \frac{\alpha-\gamma}{\beta}$

$\Leftrightarrow (tanA - m)(tanB - n) = 0$ when $\alpha^2 - \gamma^2 = \beta^2$ (1)

ie when $\alpha^2 = \beta^2 + \gamma^2$

[The official solution shows that, for "if and only if" proofs, it may be sufficient to indicate that the line of reasoning is reversible (assuming that this is the case).]

(i) With $\alpha = 2, \beta = \sqrt{3}$ & $\gamma = 1, A = x$ & $B = \frac{\pi}{4}, \beta^2 + \gamma^2 = \alpha^2$,

so that, unless $cosx = 0$,

$2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}\cos\left(x + \frac{\pi}{4}\right) = \sin\left(x + \frac{\pi}{4}\right)$

$\Leftrightarrow \left(tanx + \sqrt{3}\right)\left(tan\left(\frac{\pi}{4}\right) - \frac{1}{\sqrt{3}}\right) = 0$, from (1),

$\Leftrightarrow tanx = -\sqrt{3}$

$\Leftrightarrow x = -\frac{\pi}{3} + \pi \;\; or \;\; -\frac{\pi}{3} + 2\pi$ (given that $0 \leq x < 2\pi$);

ie the required values of x are $\frac{2\pi}{3}$ & $\frac{5\pi}{3}$ (subject to the caveat concerning $cosx$)

[Note that the double-headed arrow is important, because the statement $tanx = -\sqrt{3} \Rightarrow x = -\frac{\pi}{3} + \pi \;\; or \;\; -\frac{\pi}{3} + 2\pi$

only means "x belongs to the set $\{-\frac{\pi}{3} + \pi \,, -\frac{\pi}{3} + 2\pi\}$";

ie it could (conceivably) be the case that x can never equal $-\frac{\pi}{3} + 2\pi$ (for example)]

If $cosx = 0$, then $x = \frac{\pi}{2} \;\; or \;\; \frac{3\pi}{2}$ (given that $0 \leq x < 2\pi$).

For $x = \frac{\pi}{2}$,

$2\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \sqrt{3}\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$= 2 \,.\frac{1}{\sqrt{2}} + \sqrt{3}\,\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} \neq 0$

For $x = \frac{3\pi}{2}$,

$2\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right) + \sqrt{3}\cos\left(\frac{3\pi}{2} + \frac{\pi}{4}\right) - \sin\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)$

$= 2\,\left(-\frac{1}{\sqrt{2}}\right) + \sqrt{3}\,\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} \neq 0$;

and so we can exclude cases where $cosx = 0$;

ie the required values are $x = \frac{2\pi}{3}$ and $\frac{5\pi}{3}$

(ii) With $\alpha = 2, \beta = \sqrt{3}$ & $\gamma = 1, A = x$ & $B = \frac{\pi}{6}, \beta^2 + \gamma^2 = \alpha^2$, so that the equation $\Leftrightarrow \left(tanx+\sqrt{3}\right)\left(tan\left(\frac{\pi}{6}\right)-\frac{1}{\sqrt{3}}\right)=0$,

unless $cosx=0$ (when $x=\frac{\pi}{2}\ or\ \frac{3\pi}{2}$)

As $tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$, the original equation is true for all values of x, apart from $x=\frac{\pi}{2}\ or\ \frac{3\pi}{2}$ possibly.

For $x=\frac{\pi}{2}$,

$2\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)+\sqrt{3}\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)$

$=2\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\left(-\frac{1}{2}\right)-\frac{\sqrt{3}}{2}=0$

For $x=\frac{3\pi}{2}$,

$2\sin\left(\frac{3\pi}{2}-\frac{\pi}{6}\right)+\sqrt{3}\cos\left(\frac{3\pi}{2}+\frac{\pi}{6}\right)-\sin\left(\frac{3\pi}{2}+\frac{\pi}{6}\right)$

$=2\left(-\frac{\sqrt{3}}{2}\right)+\sqrt{3}\left(\frac{1}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)=0$

Thus the original equation is true for all values of x.

(iii) [Forcing the LHS into the earlier form:]

Let $A=\frac{1}{2}\left(\left[x+\frac{\pi}{3}\right]+3x\right)=2x+\frac{\pi}{6}$,

so that $B=3x-\left(2x+\frac{\pi}{6}\right)=x-\frac{\pi}{6}$

Then with $\alpha=2,\beta=\sqrt{3}\ \&\ \gamma=1$ again,

$\left(\tan\left(2x+\frac{\pi}{6}\right)+\sqrt{3}\right)\left(tan\left(x-\frac{\pi}{6}\right)-\frac{1}{\sqrt{3}}\right)=0$ (2),

unless $\cos\left(2x+\frac{\pi}{6}\right)=0$ or $\cos\left(x-\frac{\pi}{6}\right)=0$ (considered later)

Then $0\le x<2\pi \Rightarrow \frac{\pi}{6}\le 2x+\frac{\pi}{6}<4\pi+\frac{\pi}{6}$

and $-\frac{\pi}{6} \leq x - \frac{\pi}{6} < 2\pi - \frac{\pi}{6}$

Then (2) $\Leftrightarrow 2x + \frac{\pi}{6} = -\frac{\pi}{3} + \pi$ or $-\frac{\pi}{3} + 2\pi$ or $-\frac{\pi}{3} + 3\pi$

or $-\frac{\pi}{3} + 4\pi$ or $x - \frac{\pi}{6} = \frac{\pi}{6}$ or $\frac{\pi}{6} + \pi$

ie the possible values for x are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, $\frac{\pi}{3}$ or $\frac{4\pi}{3}$

Considering the possibility that $\cos\left(2x + \frac{\pi}{6}\right) = 0$:

Write $u = 2x + \frac{\pi}{6}$, so that $\frac{\pi}{6} \leq u < 4\pi + \frac{\pi}{6}$,

and $cosu = 0 \Leftrightarrow u = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{2} + 2\pi$, $\frac{3\pi}{2} + 2\pi$ (*)

Then, as $x = \frac{1}{2}\left(u - \frac{\pi}{6}\right)$,

$(*) \Leftrightarrow x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Considering $f(x) = 2\sin\left(x + \frac{\pi}{3}\right) + \sqrt{3}\cos(3x) - \sin(3x)$:

$f\left(\frac{\pi}{6}\right) = 2 + 0 - 1 \neq 0$

$f\left(\frac{2\pi}{3}\right) = 0 + \sqrt{3} - 0 \neq 0$

$f\left(\frac{7\pi}{6}\right) = -2 + 0 + 1 \neq 0$

$f\left(\frac{5\pi}{3}\right) = 0 - \sqrt{3} - 0 \neq 0$

Considering the possibility that $\cos\left(x - \frac{\pi}{6}\right) = 0$:

Write $u = x - \frac{\pi}{6}$, so that $-\frac{\pi}{6} \leq u < 2\pi - \frac{\pi}{6}$, and $cosu = 0 \Leftrightarrow u = \frac{\pi}{2}, \frac{3\pi}{2}$ (**)

Then, as $x = u + \frac{\pi}{6}$,

$(**) \Leftrightarrow x = \frac{2\pi}{3}\ \ or\ \ \frac{5\pi}{3}$,

both of which have already been rejected.

So we can exclude the case where $\cos\left(2x + \frac{\pi}{6}\right) = 0$.