STEP 2007, Paper 2, Q2 – Solution (4 pages; 28/3/25)

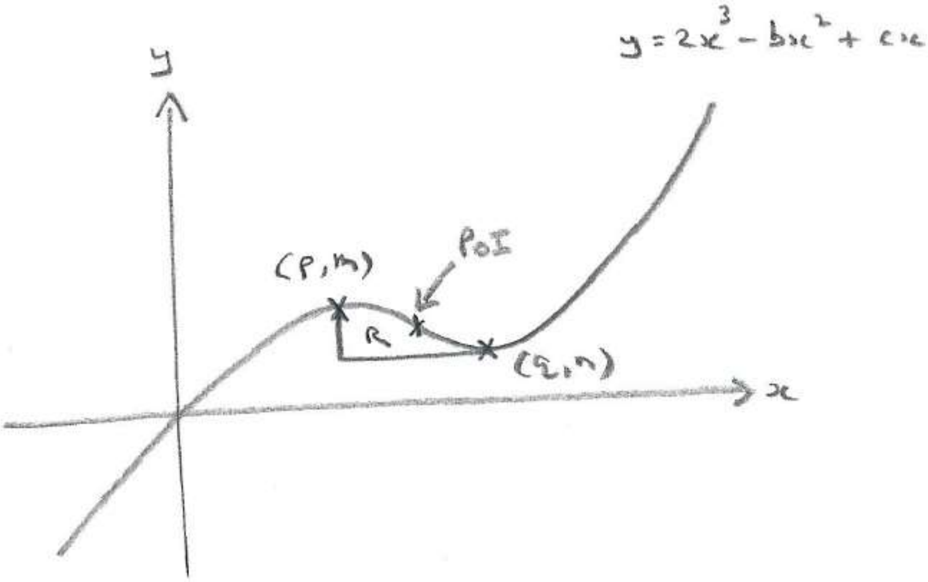
2 A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

(i) Express b and c in terms of p and q.

(ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R. Describe the symmetry of the curve.

(iii) Show that $m - n = (q - p)^3$.

(iv) Show that the area of R is $\frac{1}{2}(q - p)^4$.



(i) $y = 2x^3 - bx^2 + cx$

$\frac{dy}{dx} = 6x^2 - 2bx + c$

Turning point $\Rightarrow 6x^2 - 2bx + c = 0$ (1)

The roots of (1) are p & q,

so that $p + q = \frac{2b}{6}$ & $pq = \frac{c}{6}$

Hence $b = 3(p + q)$ & $c = 6pq$

(ii) The point of inflexion is the turning point of the gradient;

ie where $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$

So $\frac{d^2y}{dx^2} = 12x - 2b = 0$, and hence $x = \frac{b}{6} = \frac{p+q}{2}$

ie the point of inflexion lies halfway between the turning points.

There is rotational symmetry (of order 2) about the point of inflexion.

[All cubics have a single point of inflexion, and it always lies halfway between the turning points (if they exist). It isn't clear whether a proof of the symmetry is required for this question. It would be strange for no justification to be needed, but I would be surprised if the following was intended. See "Cubic Functions" for an alternative method.]

To demonstrate rotational symmetry about $x = \frac{b}{6}$, we require that

$f\left(\frac{b}{6} + t\right) - f\left(\frac{b}{6}\right) = -\{f\left(\frac{b}{6} - t\right) - f\left(\frac{b}{6}\right)\}$;

ie that $f\left(\frac{b}{6} + t\right) + f\left(\frac{b}{6} - t\right) = 2f\left(\frac{b}{6}\right)$

where $f(x) = 2x^3 - bx^2 + cx$

Now $f\left(\frac{b}{6}\right) = \frac{2b^3}{6^3} - \frac{b^3}{36} + \frac{bc}{6}$

and $f\left(\frac{b}{6} + t\right) + f\left(\frac{b}{6} - t\right) = 2(\frac{b}{6} + t)^3 - b\left(\frac{b}{6} + t\right)^2 + c\left(\frac{b}{6} + t\right)$

$+2(\frac{b}{6} - t)^3 - b\left(\frac{b}{6} - t\right)^2 + c\left(\frac{b}{6} - t\right)$

$= \frac{4b^3}{6^3} + 4(3)\left(\frac{b}{6}\right)t^2 - \frac{2b^3}{36} - 2bt^2 + \frac{2bc}{6}$

$= \frac{4b^3}{6^3} - \frac{2b^3}{36} + \frac{2bc}{6} = 2f\left(\frac{b}{6}\right)$, as required

(iii) $m - n = f(p) - f(q)$

$= 2(p^3 - q^3) - b(p^2 - q^2) + c(p - q)$

$= 2(p-q)(p^2+pq+q^2) - 3(p+q)(p^2-q^2) + 6pq(p-q)$

$= (p-q)\{2p^2+2pq+2q^2 - 3(p+q)^2 + 6pq\}$

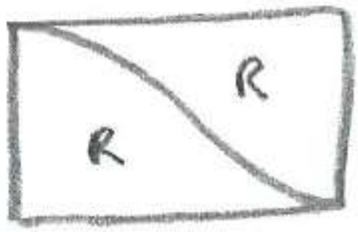
$= (p-q)\{2(p+q)^2 - 2pq - 3(p+q)^2 + 6pq\}$

$= (p-q)\{4pq - (p+q)^2\}$

$= -(p-q)(p-q)^2$

$= (q-p)^3$, as required

(iv)



$R = 1/2 \times$ area of rectangle in diagram above (by symmetry of the cubic about the point of inflexion)

$= \frac{1}{2}(q-p)(m-n)$

$= \frac{1}{2}(q-p)(q-p)^3$

$= \frac{1}{2}(q-p)^4$, as required