STEP 2007, Paper 1, Q4 – Solution (7 pages; 21/5/18)

[The Factor theorem could be used to demonstrate that $x + b + c$ is a factor of $f(x) = x^3 - 3xbc + b^3 + c^3$:

$f(-b - c) = -(b + c)^3 + 3(b + c)bc + b^3 + c^3$

$= 3(b + c)bc - (3b^2c + 3bc^2) = 0$

However, looking ahead in the question, we can see that $Q(x)$ is probably needed.]

Method 1

Suppose that $x^3 - 3xbc + b^3 + c^3 = (x + b + c)(x^2 + px + q)$

Using the fact that $b^3 + c^3 = (b + c)(b^2 - bc + c^2)$,

[This result, together with its companion $b^3 - c^3 = (b - c)(b^2 + bc + c^2)$, is very popular with the STEP examiners.]

we see that $q = b^2 - bc + c^2$

[By observing that $x^3 - 3xbc + b^3 + c^3$ is symmetric in x, b & c, we could in fact surmise at this stage that $p = -(b + c)$, to give

$Q(x) = x^2 + b^2 + c^2 - bc - bx - cx$]

Equating coefficients of x^2: $0 = b + c + p$, giving $p = -(b + c)$

[Also, had we not used the factorisation of $b^3 + c^3$,

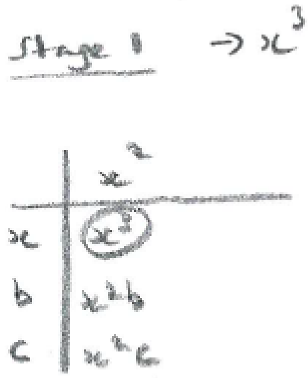
equating coefficients of x: $-3bc = q + (b + c)p$,

so that $q = -3bc + (b + c)^2 = b^2 - bc + c^2$]

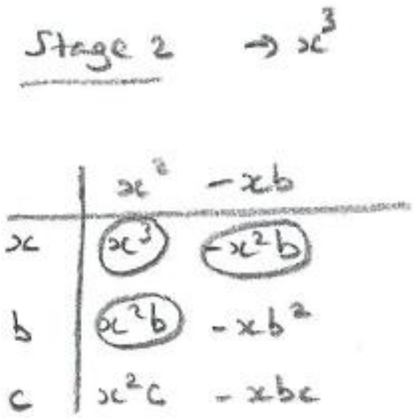
Method 2

[In the official solutions, it is suggested that this sort of factorisation can be done in your head. The following table approach is perhaps a bit safer though.]

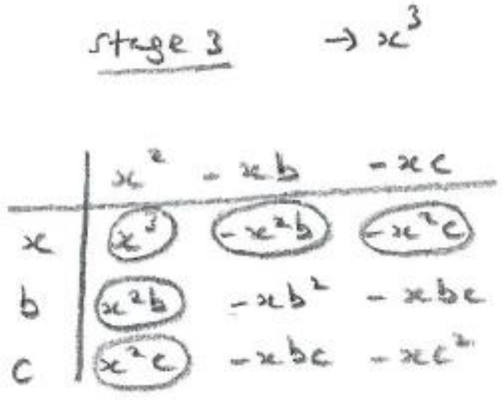
First of all, we put an x^2 in the 1st position in the top row, as this will generate the x^3 needed, when multiplied by the x. The x^3 is placed in the table and ringed, to show that it has been processed. Multiplying the x^2 by b & c in turn produces the terms x^2b & x^2c. These two terms are left un-ringed for the moment (see "Stage 1" below).



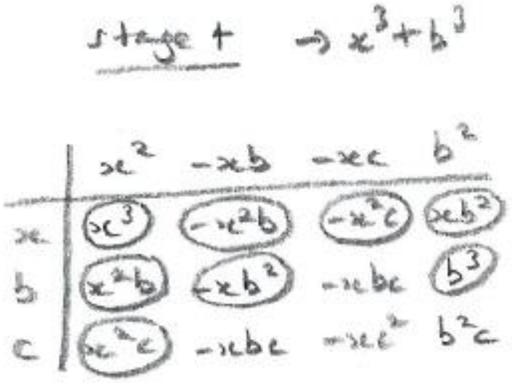
The 2nd position in the top row is then determined by the need to balance the x^2b term (noting that no such terms are required in the target expression $x^3 - 3xbc + b^3 + c^3$). Thus we place a $-xb$ in the top row, as this gives $-x^2b$ when multiplied by the x. The two balancing terms can then be ringed, to show that they have been processed. Multiplying the $-xb$ by b & c in turn produces the terms $-xb^2$ & $-xbc$. Note that the $-xbc$ is part of the target expression (see "Stage 2" below).



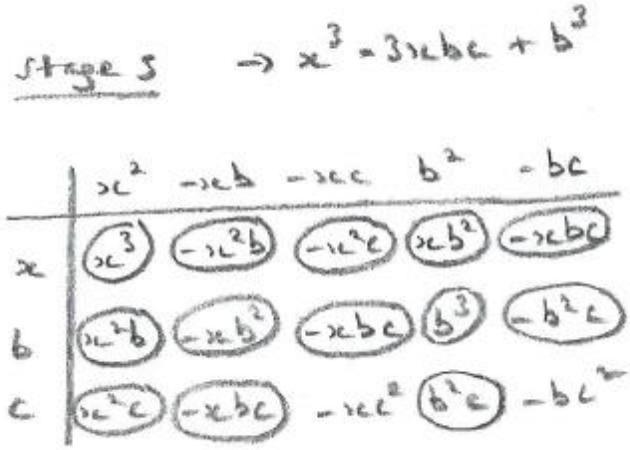
$-xc$ then goes in the 3rd position in the top row, in order to eliminate the x^2c (see "Stage 3" below).



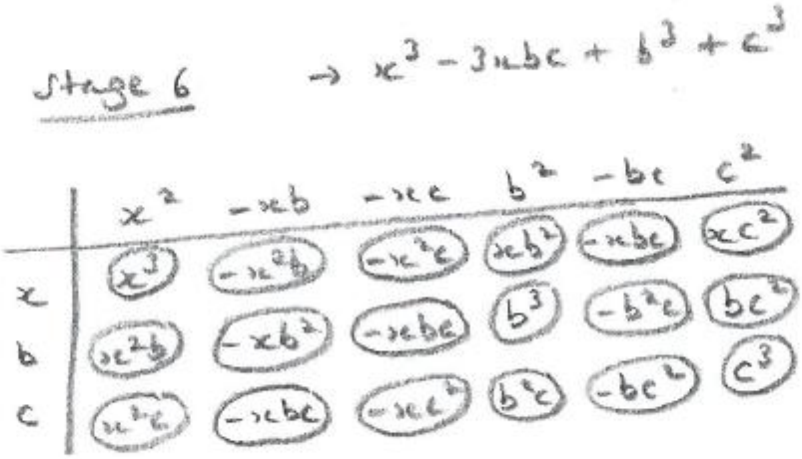
hen the next term to be eliminated is $-xb^2$, which requires a b^2 in the top row. This generates the term b^3, which can be ringed, as it is needed for the target expression (see "Stage 4" below).



The next unringed term is $-xbc$. As $-3xbc$ is needed for the target expression, we add a $-bc$ to the top row, to enable 3 terms of $-xbc$ to be ringed, as well as the balancing terms b^2c & $-b^2c$. This leaves two un-ringed terms: $-xc^2$ & $-bc^2$ (see "Stage 5" below).



In order to eliminate the $-xc^2$, we add a c^2 to the top row, and this also gets rid of the $-bc^2$, and gives the final c^3 needed for the target expression (see "Stage 6" below).



Method 3

Noting that the expression $x^3 - 3xbc + b^3 + c^3$ is symmetric in x, b & c, and that $Q(x)$ will need to have an x^2 term, it follows that it must also include the terms b^2 & c^2. Observing then that the expansion of

$(x + b + c)(x^2 + b^2 + c^2)$ produces the unwanted term bx^2 (for example), the simplest possible adjustment needed is to add in the term $-bx$ to $Q(x)$, along with $-cx$ & $-bc$ (by symmetry).

$(x+b+c)(x^2+b^2+c^2-bx-cx-bc)$ can then be seen to expand to give $x^3-3xbc+b^3+c^3$, as required.

To write $2Q(x)$ as a sum of 3 perfect squares, we note that $(b-c)^2$ (for example) accounts for some of the terms in

$2Q(x) = 2x^2+2b^2+2c^2-2bc-2bx-2cx$, and that (since this expression is symmetric in x, b & c), we would expect to use

$(x-b)^2$ & $(x-c)^2$ as well (noting that there is no asymmetry, since $(b-c)^2 = (c-b)^2$)

Thus $2Q(x) = (b-c)^2 + (x-b)^2 + (x-c)^2$

For the next part, we are told that $ak^2+bk+c = bk^2+ck+a = 0$ (A)

[As the result to be proved doesn't involve k^2, we could try eliminating it from the above equations.]

Hence $-k^2 = \frac{bk+c}{a}$, and also $-k^2 = \frac{ck+a}{b}$ (since $a, b \neq 0$)

Thus $\frac{bk+c}{a} = \frac{ck+a}{b}$, and so $b^2k+bc = ack+a^2$

and hence $(ac-b^2)k = bc-a^2$, as required. (B)

[It is tempting now to apply the same method to derive an equation involving k^2 (but not k). However, this would involve dividing by c, and we are not told that $c \neq 0$.]

As $ac \neq b^2$, the preceding result $\Rightarrow k = \frac{bc-a^2}{ac-b^2}$

Substitution into the 2nd term of the 1st half of (A) then gives

$ak^2 + b(\frac{bc-a^2}{ac-b^2}) + c = 0$

$\Rightarrow ak^2(ac - b^2) + b(bc - a^2) + c(ac - b^2) = 0$

$\Rightarrow ak^2(ac - b^2) = ba^2 - cac$

$\Rightarrow (ac - b^2)k^2 = ab - c^2$ (C)

[which is in a similar form to the result involving k]

Squaring both sides of (B) then gives

$(ac - b^2)^2k^2 = (bc - a^2)^2$, which combined with (C) gives

$(ac - b^2)(ab - c^2) = (bc - a^2)^2$, as required.

[We then have to trust that the next result follows on directly from this (rather than from the earlier results). Alternatively, we could reason (to ourselves) as follows: the final deduction (that either $k = 1$ or the two equations are identical) follows from the $(x + b + c)Q(x)$ result. So, if

$a^3 - 3abc + b^3 + c^3 = 0$, *didn't* follow directly from

$(ac - b^2)(ab - c^2) = (bc - a^2)^2$, no use would have been made of the latter result.]

From this result, $a^2bc - ac^3 - ab^3 + b^2c^2 = b^2c^2 - 2bca^2 + a^4$

and so $a^2bc - ac^3 - ab^3 + 2bca^2 - a^4 = 0$

As $a \neq 0$, $abc - c^3 - b^3 + 2bca - a^3 = 0$

or $a^3 - 3abc + b^3 + c^3 = 0$, as required

So, from the earlier expression established for $Q(x)$,

$(a+b+c).\frac{1}{2}\{(b-c)^2+(a-b)^2+\ (a-c)^2\}=0$ (D)

Thus either $a+b+c=0$ or $b-c=a-b=a-c=0$ (ie $a=b=c$)

In the former case, the two equations both have roots 1 and k (if $k\neq 1$).

Then either $k=1$ or the two equations have the same roots: 1 and k, and are hence identical (apart from a multiplier).

[The official solutions just say that $a+b+c=0\Rightarrow k=1$. This may not be a correct deduction.]

In the latter case, the two equations are identical.

[As you can see, there is a lot to do for this question: 7 tasks in total!]