STEP 2007, Paper 1, Q2 – Solution (2 pages; 21/5/18)

(i) $\tan(A+B) = \frac{tanA+tanB}{1-tanAtanB} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\left(\frac{5}{6}\right)}{\left(\frac{5}{6}\right)} = 1$

so that $A + B = tan^{-1}(1) = \frac{\pi}{4}$, since $0 < A + B < \frac{\pi}{2} + \frac{\pi}{2} = \pi$

[For the next part, note that $p = 2$ & $q = 3$ are solutions, but not necessarily the only ones.]

Let $P = arctan\left(\frac{1}{p}\right)$ & $Q = arctan\left(\frac{1}{q}\right)$

[If in doubt, create some letters and set up some equations.]

Then $P + Q = \frac{\pi}{4}$, and hence $\tan(P + Q) = 1$

Also, $\tan(P+Q) = \frac{tanP+tanQ}{1-tanPtanQ} = \frac{\frac{1}{p}+\frac{1}{q}}{1-\left(\frac{1}{p}\right)\left(\frac{1}{q}\right)} = \frac{\left(\frac{p+q}{pq}\right)}{\left(\frac{pq-1}{pq}\right)} = \frac{p+q}{pq-1}$

Hence $\frac{p+q}{pq-1} = 1$ and $p + q = pq - 1$,

so that $(p-1)(q-1) = pq - (p+q) + 1 = [p + q + 1] - (p+q) + 1 = 2$,

as required.

Then, as p & q $\in \mathbb{Z}$ (excluding 0),

either $p - 1 = 2$ & $q - 1 = 1$ (or v.v.)

in which case $p = 3$ & $q = 2$ (or v.v.);

or $p - 1 = -2$ & $q - 1 = -1$ (or v.v.),

in which case $p = -1$ & $q = 0$ (contradiction, as $p, q \neq 0$)

Thus, $p = 3$ & $q = 2$, or v.v.

[My version of the official solution says "Since p & q are positive integers ..." (whereas the question says that p & q are non-zero integers), and so doesn't consider the possibility of $p \; or \; q$ being negative.]

(ii) The working in (i) did not initially depend on p & q being integers,

so, by making the substitutions $p = r$ & $q = \frac{s+t}{s}$,

$(p-1)(q-1) = 2 \Rightarrow (r-1)\left(\frac{t}{s}\right) = 2 \Rightarrow (r-1)t = 2s$

Then, as t & s have no common factors, and t is a divisor of $2s$,

either $t = 2$ or $t = 1$ (given that $t \in \mathbb{Z}^+$).

If $t = 2$, then $r - 1 = s$, so that $r = s + 1$

If $t = 1$, then $r - 1 = 2s$, so that $r = 2s + 1$

[This last step doesn't seem very demanding. The official solution goes on to give a related result, but this isn't asked for in the question. It also mentions that s must be odd if $t = 2$ (since s & t have no common factors), but again this doesn't seem to be needed.]