STEP 2007, Paper 1, Q12 – Solution (3 pages; 21/5/18)

(i) $P(1st\ is\ R) = \frac{a}{N}$

$P(2nd\ is\ R) = P(1st\ is\ R)P(2nd\ is\ R|1st\ is\ R)$

$+P(1st\ is\ not\ R)P(2nd\ is\ R|1st\ is\ not\ R)$

$= \left(\frac{a}{N}\right)\left(\frac{a-1}{N-1}\right) + \left(\frac{N-a}{N}\right)\left(\frac{a}{N-1}\right)$

$= \frac{a}{N(N-1)}(a-1+N-a) = \frac{a}{N} = P(1st\ is\ R)$, as required

[We could also argue along the following lines: "Drawing one sweet and then another is no different from putting both hands into the bag and drawing a sweet with each hand, but designating the right-hand sweet as the 1st drawn. But alternatively we could have designated the right-hand sweet as the 2nd drawn." However, it is hard to be sure that a 'convincing' argument has been made (especially if it doesn't appear in the examiners' mark scheme), so the calculation is probably safer.]

(ii) The examiners' report strongly recommends the use of a tree diagram. Alternatively, you can just break the problem down into separate cases, as below.

$P(1st\ is\ R) = p\left(\frac{a}{N}\right) + q\left(\frac{b}{N}\right) = \frac{pa+qb}{N}$

Case 1: If 1st coin is H

$P(2nd\ [sweet]\ is\ R) = P(1st\ is\ R)P(2nd\ is\ R|1st\ is\ R)$

$+P(1st\ is\ not\ R)P(2nd\ is\ R|1st\ is\ not\ R)$

$= \frac{a}{N}\{p\left(\frac{a-1}{N-1}\right) + q\left(\frac{b+1}{N+1}\right)\} + \frac{(N-a)}{N}\{p\left(\frac{a}{N-1}\right) + q\left(\frac{b}{N+1}\right)\}$

$= \frac{1}{N(N-1)(N+1)}\{ap(a-1)(N+1) + aq(b+1)(N-1)$

$+(N-a)pa(N+1)+(N-a)qb(N-1)\}$

$= \frac{1}{N(N-1)(N+1)}\{ap(N+1)[a-1+N-a]$

$+q(N-1)[ab+a+Nb-ab]\}$

$= \frac{1}{N(N-1)(N+1)}\{ap(N+1)(N-1)+q(N-1)(a+Nb)\}$

$= \frac{1}{N(N+1)}\{apN+ap+qa+qNb\}$

$= \frac{1}{N(N+1)}\{N(ap+qb)+a\}$, since $p+q=1$

Case 2: If 1st coin is T

The situation is the same as before, with the roles of a & b exchanged, and also the roles of p & q.

Thus $P(2nd\ [sweet]\ is\ R) = \frac{1}{N(N+1)}\{N(bq+pa)+b\}$

Finally, $P(2nd\ is\ R) = P(1st\ coin\ is\ H)P(2nd\ is\ R|1st\ coin\ is\ H)$

$+P(1st\ coin\ is\ T)P(2nd\ is\ R|1st\ coin\ is\ T)$

$= \frac{p}{N(N+1)}\{N(ap+qb)+a\} + \frac{q}{N(N+1)}\{N(bq+pa)+b\}$

$= \frac{1}{N(N+1)}\{pN(ap+qb)+pa+qN(bq+pa)+qb\}$

$= \frac{1}{N(N+1)}\{(pN+qN)(ap+qb)+pa+qb\}$

$= \frac{1}{N(N+1)}\{N(ap+qb)+pa+qb\}$, as $p+q=1$

$= \frac{1}{N(N+1)}\{(N+1)(ap+qb)\}$

$= \frac{pa+qb}{N} = P(1st\ is\ red)$, as required