STEP 2006, Paper 3, Q4 - Solution (2 pages; 19/5/18)

Let $x = y$, to give $2f(x) \equiv f(2x)$ (1)

Let $u = 2x$. Then, differentiating both sides wrt x:

$$2f'(x) = \frac{d}{du}f(u).\frac{du}{dx} = f'(u)(2) = 2f'(2x);\text{ ie } f'(x) = f'(2x)$$

Then $f''(x) = \frac{d}{dx}f'(2x) = \frac{d}{du}f'(u).\frac{du}{dx} = f''(u)(2) = 2f''(2x)$

So $f''(0) = 2f''(0)$, and hence $f''(0) = 0$, as required.

From $f''(x) = 2f''(2x)$, $f^{(3)}(x) = 2\frac{d}{du}f''(u)\frac{du}{dx} = 4f^{(3)}(2x)$,

and so on for higher derivatives, so that $f^{(n)}(0) = 0$ for $n \geq 2$

Also from (1), $2f(0) \equiv f(0)$, so that $f(0) = 0$ as well.

The Maclaurin series for $f(x)$ is

$f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \cdots$, so that in this case

$f(x) = xf'(0) = ax$, say, where a is a constant for a given $f(x)$

(i) $g(x)g(y) = g(x+y) \Rightarrow lng(x) + lng(y) = lng(x+y)$

ie $G(x) + G(y) = G(x+y) \Rightarrow G(x) = ax$

ie $lng(x) = ax$, so that $g(x) = e^{ax}$

(ii) $h(x) + h(y) = h(xy) \Rightarrow h(e^u) + h(e^v) = h(e^u e^v) = h(e^{u+v})$

$\Rightarrow H(u) + H(v) = H(u+v) \Rightarrow H(x) = ax$

ie $h(e^x) = ax$

Let $z = e^x$; then $h(z) = alnz$

(iii) The form of $z = \frac{x+y}{1-xy}$ suggests letting $x = tan\theta$ & $y = tan\phi$, when $z = \tan(\theta + \phi)$

Then $t(x) + t(y) = t(z)$ [with t presumably hinting at tan!]

$\Rightarrow t(tan\theta) + t(tan\phi) = t(\tan(\theta + \phi))$

Let $T(x) = t(tanx)$

so that $T(\theta) + T(\phi) = T(\theta + \phi)$

$\Rightarrow T(x) = ax$; ie $t(tanx) = ax$

Let $u = tanx$; then $t(u) = a(arctanu)$