STEP 2006, Paper 3, Q10 – Solution (2 pages; 19/5/18)

Conservation of energy:

$\frac{1}{2}mV^2 + \frac{1}{2}mk^2\Omega^2 + \frac{1}{2}ma^2\Omega^2 = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mk^2\omega^2 + \frac{1}{2}mr^2\omega^2$

(ma^2 & mr^2 being the moment of inertia of the particle initially and at time t)

$\Rightarrow V^2 + k^2\Omega^2 + a^2\Omega^2 = \left(\frac{dr}{dt}\right)^2 + k^2\omega^2 + r^2\omega^2$

$\Rightarrow \left(\frac{dr}{dt}\right)^2 = V^2 + \Omega^2(k^2 + a^2) - \omega^2(k^2 + r^2)$

[noting the forms of the expressions to be established]

Also, from conservation of angular momentum:

$(mk^2 + ma^2)\Omega = (mk^2 + mr^2)\omega$

$\Rightarrow \omega = \frac{\Omega(k^2+a^2)}{k^2+r^2}$, as required (1)

Then $\left(\frac{dr}{dt}\right)^2 = \frac{\Omega^2 a^2(k^2+a^2)}{k^2} + \Omega^2(k^2 + a^2) - \frac{\Omega^2(k^2+a^2)^2}{k^2+r^2}$

$= \frac{\Omega^2(k^2+a^2)}{k^2(k^2+r^2)}\{a^2(k^2 + r^2) + k^2(k^2 + r^2) - k^2(k^2 + a^2)\}$

$= \frac{\Omega^2(k^2+a^2)}{k^2(k^2+r^2)}\{r^2(a^2 + k^2)\} = \frac{\Omega^2 r^2(k^2+a^2)^2}{k^2(k^2+r^2)}$, as required (2)

$\frac{dr}{d\theta} = \frac{dr}{dt}.\frac{dt}{d\theta} = \frac{dr}{dt}.\frac{1}{\omega}$

Then from (1) & (2), $\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{\omega^2}\left(\frac{dr}{dt}\right)^2 = \left(\frac{k^2+r^2}{\Omega(k^2+a^2)}\right)^2 \frac{\Omega^2 r^2(k^2+a^2)^2}{k^2(k^2+r^2)}$

$= \frac{r^2(k^2+r^2)}{k^2}$

As the particle is moving towards 0, $\frac{dr}{dt} < 0$, and hence

$\frac{dr}{d\theta} = \frac{dr}{dt} . \frac{1}{\omega} < 0$

So $\frac{dr}{d\theta} = -\frac{r(k^2+r^2)^{1/2}}{k}$, giving $k\frac{dr}{d\theta} = -r(k^2 + r^2)^{\frac{1}{2}}$, as required.

$\frac{d\theta}{dr} = -\frac{k}{r\sqrt{k^2+r^2}}$ and hence $\theta - 0 = -k\int_a^r \frac{1}{R\sqrt{k^2+R^2}} dR$

[as an alternative to establishing the constant of integration explicitly; R is a dummy variable]

Let $u = k/R$, so that $du = -\frac{k}{R^2} dR$

and $\theta = \int_{k/a}^{k/r} \frac{u}{\sqrt{k^2+R^2}} \left(\frac{R^2}{k}\right) du = \int_{k/a}^{k/r} \frac{u}{\sqrt{u^2+1}} \left(\frac{R}{k}\right) du$

$= \int_{\frac{k}{a}}^{\frac{k}{r}} \frac{1}{\sqrt{u^2+1}} du = [arsinhu]_{k/a}^{k/r}$

So $\theta = arsinh\left(\frac{k}{r}\right) - arsinh(\frac{k}{a})$

and $\theta + \alpha = arsinh\left(\frac{k}{r}\right)$, where $sinh\alpha = k/a$

Thus $rsinh(\theta + \alpha) = k$, as required.

The particle reaches the axis when $r = 0$, which requires $\theta = \infty$; ie it never happens.