STEP 2006, Paper 3 – Notes (3 pages; 20/5/18)

See separate documents for Sol'ns.

1	2	3	4	5	6	7	8
N	N	Sol'n	Sol'n	N	N	N	

9	10	11		12	13	14
	Sol'n	Sol'n		Sol'n	Sol'n	

Q1 Note in (iii) that $y^2 = x^2 + 1$ can be written as $\frac{y^2}{1} - \frac{x^2}{1} = 1$, which is a hyperbola with asymptotes y $= \pm\frac{1}{1}x$, passing through (0,1) & (0,-1) (ie with directrices parallel to the x axis)

Q2 In (ii), note that $sec^2\theta$ is the derivative of something lurking elsewhere in the integrand – namely tanθ, and that, with the substitution u = tanθ, $\frac{1}{1+(ucos2\alpha)^2}$ can be integrated. In general, it is usually easiest to spot the derivative of u in the integrand, since it will always be on the top.

In the last part, you may find that you had overlooked the significance of $0<\alpha<\frac{1}{4}\pi$ earlier on (it might be necessary to go back and allow for it). It is tempting to think (as I did) that the crux of the last part is that the integral of $\frac{1}{sec^2\alpha+u^2}$ has a different form if sec $\alpha < 0$, but this isn't the case: it is just that the limits of the integral have now changed.

Q5 As an alternative to the method described in the Hints & Answers, the 1st part can be tackled by establishing that $\arg(\frac{\gamma-\alpha}{\beta-\alpha}) = \arg(\frac{\gamma-\beta}{\gamma-\alpha})$ & hence that $\frac{\gamma-\alpha}{\beta-\alpha} = k\frac{\gamma-\beta}{\gamma-\alpha}$, with $k = 1$

The Hints & Answers demonstrate the useful technique of combining the two sol'ns a=b & c=d into the single e'qn

(a-b)(c-d)=0

Q6 Either the syllabus has changed, or this question is rather unfair, as it assumes knowledge of the polar form of the equation of a parabola, and the fact that the origin is taken to be the focus of the parabola for this purpose (for the Cartesian form $y^2 = 4ax$, the focus is at $(0, a)$, with the parabola also being reflected in the y-axis). [Although there is a clue, as the focus is labelled O!]

When it comes to solving the equations

$$\tan(\frac{\theta}{2}) = -\frac{\frac{dr}{d\theta}-rtan\theta}{\frac{dr}{d\theta}tan\theta+r} \quad \text{and} \quad tan\theta = \frac{2tan\left(\frac{\theta}{2}\right)}{1-tan^2\left(\frac{\theta}{2}\right)}$$

there are two instructive points to note:

(i) the use of a substitution such as $t = tan\left(\frac{\theta}{2}\right)$ [a standard substitution - at least, as far as the examiners are concerned]

to simplify the working

(ii) considering more than one option before proceeding with the algebra: thus, substituting for tan $\left(\frac{\theta}{2}\right)$ in the 2nd equation leads nowhere, as it involves non-linear terms for $\frac{dr}{d\theta}$ (which needs to be made the subject of the equation); whereas substituting for $tan\theta$ in the 1st equation avoids this problem

Q7 In (i) of the "Hints & Answers", "(or $-coshx \pm sinhx$)" should be "(or $-sinhx \pm coshx$)", I believe.

For (ii), there are (at least) two plausible approaches:

(a) try to rearrange the DE into the form in (i); eg by making the substitution z = coshy (as suggested by the required form of the sol'n)

This doesn't in fact work, though the resulting DE can be solved by approach (b).

(b) simply apply the same method as for (i); ie solve the quadratic in dy/dx, and then solve the resulting 1st order DE

At the end of (ii), in the "Hints & Answers", the printing error makes it unclear what the method is, but it seems that you can work backwards from the asymptote, showing that when

y = ±(-x + ln4), coshy tends to $2e^{-x}$-1 as $x \to -\infty$ (noting that $2e^{-x}$-1 is undefined for x>0)

(The fact that the asymptote is of the form y = mx+c means that we are not talking about a vertical asymptote (ie x → some constant), but $x \to -\infty$).