STEP 2006, P2, Q4 – Solution (2 pages; 17/5/18)

If $x = \pi - t,\ dx = -dt\ \&\ sinx = sint,$

so that $I = \int_0^\pi xf(sinx)dx = \int_\pi^0 (\pi - t)f(sint)(-dt)$

$\pi \int_0^\pi f(sint)dt - \int_0^\pi tf(sint)dt$

Hence $2I = \pi \int_0^\pi f(sint)dt,$ and $= \frac{\pi}{2}\int_0^\pi f(sinx)dx,$ as required

(i) By the above result, $\int_0^\pi \frac{xsinx}{3+sin^2x}dx = \frac{\pi}{2}\int_0^\pi \frac{sinx}{3+sin^2x}dx$

$= \frac{\pi}{2}\int_0^\pi \frac{sinx}{4-cos^2x}dx$

Let $u = cosx$, so that $du = -sinx\ dx$

and $\frac{\pi}{2}\int_0^\pi \frac{sinx}{4-cos^2x}dx = \frac{\pi}{2}\int_1^{-1} \frac{-1}{(2-u)(2+u)}du$

$= \frac{\pi}{2}\int_{-1}^1 \frac{A}{2-u} + \frac{B}{2+u}du$

Then $1 = A(2+u) + B(2-u)$

$u = 2 \Rightarrow A = \frac{1}{4}\ ; u = -2 \Rightarrow B = \frac{1}{4}$

giving $\frac{\pi}{8}\int_{-1}^1 \frac{1}{2-u} + \frac{1}{2+u}du = \frac{\pi}{8}[-ln(2-u) + ln(2+u)]_{-1}^{1}$

$= \frac{\pi}{8}\{(0 + ln3) - (-ln3 + 0)\} = \frac{\pi ln3}{4}$

(ii) $I = \int_0^{2\pi} \frac{xsinx}{3+sin^2x}dx = \int_0^\pi \frac{xsinx}{3+sin^2x}dx + \int_\pi^{2\pi} \frac{xsinx}{3+sin^2x}dx$

$= \frac{\pi ln3}{4} + J,$

Let $t = x - \pi$, so that $sint = -\sin(\pi - x) = -sinx$

and $J = \int_0^\pi \frac{(t+\pi)(-sint)}{3+sin^2t}dt$

$= -\int_0^\pi \frac{tsint}{3+sin^2t}dt - \pi\int_0^\pi \frac{sint}{3+sin^2t}dt$

$= -\frac{\pi ln3}{4} - \pi \int_0^{\pi} \frac{sint}{3+sin^2t} dt$

Thus $I = -\pi \int_0^{\pi} \frac{sint}{3+sin^2t} dt = -2\left(\frac{\pi ln3}{4}\right) = -\frac{\pi ln3}{2}$

(iiii) [Rejected approaches:

(a) Remove the moduli signs, by splitting the range of integration into $\left(0, \frac{\pi}{2}\right)$ & $(\frac{\pi}{2}, \pi)$; but the substitution $u = 2x$ doesn't seem to lead anywhere.

(b) Find a result corresponding to the original one, but for the range $\left(0, \frac{\pi}{2}\right)$. This seems too complicated, and not very promising anyway.]

[Writing the integrand as a function of $sinx$:]

$\int_0^{\pi} \frac{x|\sin(2x)|}{3+sin^2x} dx = 2\int_0^{\pi} \frac{xsinx\sqrt{1-sin^2x}}{3+sin^2x} dx$

[noting that the square root sign implies the positive root]

$= \pi \int_0^{\pi} \frac{sinx\sqrt{1-sin^2x}}{3+sin^2x} dx$, by the original result

Let $u = cosx$, so that $du = -sinx\, dx$,

giving $\pi \int_1^{-1} \frac{-|u|}{4-u^2} du = \pi \int_{-1}^{1} \frac{|u|}{4-u^2} du$

$= \pi \int_{-1}^{0} \frac{-u}{4-u^2} du + \pi \int_0^1 \frac{u}{4-u^2} du$

$= \frac{\pi}{2}[\ln|4-u^2|]_{-1}^{0} - \frac{\pi}{2}[\ln|4-u^2|]_0^1$

$= \frac{\pi}{2}\{ln4 - ln3\} - \frac{\pi}{2}\{ln3 - ln4\}$

$= \pi(ln4 - ln3) = \pi ln\left(\frac{4}{3}\right)$