STEP 2006, P2, Q12 – Solution (3 pages; 17/5/18)

(i) [It looks as though an answer could be worked out fairly easily by considering relative frequencies, but the question says "using a Binomial model": is this an instruction to do the question by a specified method, or just reassurance that you are allowed to make a simplifying assumption? As is usually the case, the correct approach involves something simple: the key idea is conditional probability, with each bowler's own number of wickets having a Binomial distribution.]

Let P(a,b,c) be the probability of bowlers A,B & C obtaining a,b & c wickets respectively.

Then Prob(wicket is obtained by A |1 wicket is obtained)

$$= \frac{P(1,0,0)}{P(1,0,0)+P(0,1,0)+P(0,0,1)}$$

Now $P(1,0,0) = \{30.\frac{1}{36}.(\frac{35}{36})^{29}\}.(\frac{24}{25})^{30}.(\frac{40}{41})^{30}$

[If this was an A Level question, we would straightaway reject this approach as too complicated – given that calculators are not allowed – having established that statistical tables are no help here. However, being STEP we need to persevere a bit, to see if any cancelling is going to occur. Rather worryingly though, we are not being asked for an approximate answer. However, looking ahead we can see that the quantity

$(\frac{35}{36})^{29}.(\frac{24}{25})^{29}.(\frac{40}{41})^{29}$ is in fact going to cancel from top & bottom.]

and similarly forP(0,1,0) and P(0,0,1), giving:

$30.\frac{1}{36}.(\frac{35}{36})^{29}.(\frac{24}{25})^{30}.(\frac{40}{41})^{30}$

$\div \ \{ \ 30.\frac{1}{36}.(\frac{35}{36})^{29} \ .(\frac{24}{25})^{30}.(\frac{40}{41})^{30} \ + \ (\frac{35}{36})^{30} \ .30.\frac{1}{25}(\frac{24}{25})^{29}.$
$(\frac{40}{41})^{30}$

$+ (\frac{35}{36})^{30} \ .(\frac{24}{25})^{30}.30.\frac{1}{41}(\frac{40}{41})^{29} \ \}$

Cancelling $30\,(\frac{35}{36})^{29} \ .(\frac{24}{25})^{29}.(\frac{40}{41})^{29}$ from top & bottom gives:

$$\frac{\frac{1}{36}\cdot\frac{24}{25}\cdot\frac{40}{41}}{\frac{1}{36}\cdot\frac{24}{25}\cdot\frac{40}{41}+\frac{35}{36}\cdot\frac{1}{25}\cdot\frac{40}{41}+\frac{35}{36}\cdot\frac{24}{25}\cdot\frac{1}{41}}$$

$= \frac{24.40}{24.40+35.40+35.24} = \frac{24}{24+35+21} = \frac{24}{80} = 0.3$

(ii) Expected number of wickets $= 30(\frac{1}{36}+\frac{1}{25}+\frac{1}{41})$

[Given that the fractions in the brackets are supposed to average 1/30 approximately, the approximation is going to have to be fairly rough.]

$\approx 30\,(\frac{1}{30}+\frac{1}{25}+\frac{1}{45}) = 1 + 30\,.\frac{70}{25.45} = 1 + \frac{28}{15} \approx 3$

[The e^3 hints at the Poisson distribution, which features e^{-3}.]

The incidences of obtaining a wicket can be counted as rare [equivalent to n being relatively large, with p being relatively small], (approximately) random and independent of one another, and with (approximately) constant probability.

Thus the number of wickets in a match, X, can be treated as having an approximate Poisson distribution, with parameter equal to the expected number of wickets, 3.

Then Prob(X≥5) = 1 – {Prob(0)+Prob(1) + ... + Prob(4)}

$= 1 - e^{-3}(1 + 3 + \frac{9}{2} + \frac{27}{6} + \frac{81}{24}) \approx 1 - \frac{1}{20}(13 + \frac{27}{8})$

$= 1 - \frac{16.375}{20} = 1 - 0.81875 \approx 1/5$, as required