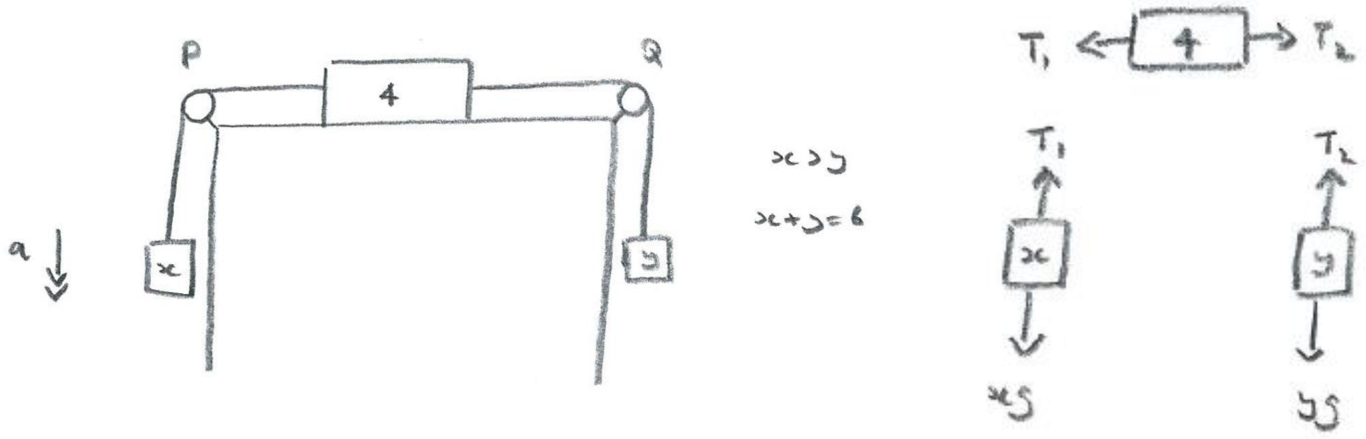
STEP 2006, Paper 1, Q9 – Solution (2 pages; 14/5/18)

Before the string is cut:



Noting that the required time is to be expressed in terms of y:

$(6-y)g - T_1 = (6-y)a \quad (1) \quad (x\ \ kg$ mass)

$T_1 - T_2 = 4a \quad (2) \quad (4kg$ mass)

$T_2 - yg = ya \quad (3) \quad (y\ kg$ mass)

Then (to find a), $(1)\&(3) \Rightarrow T_1 - T_2 = (6-y)(g-a) - y(a+g)$

Then, from (2), $4a = a(y-6-y) + 6g - yg - yg$

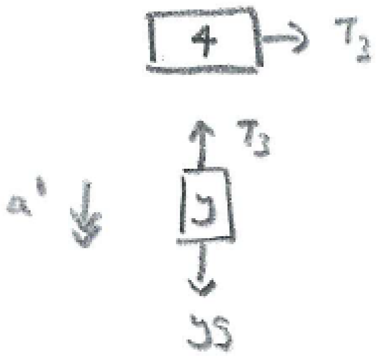
so that $10a = g(6-2y)$ & $a = \frac{g(3-y)}{5}$

If t_1 is the time until the string is cut, then the suvat equation

$s = ut + \frac{1}{2}at^2$ gives $d = \frac{1}{2}\left(\frac{g(3-y)}{5}\right){t_1}^2$,

so that ${t_1}^2 = \frac{10d}{g(3-y)}$ (4) and the speed of the 4 kg mass when the string is cut is (from "$v = u + at$") $\left(\frac{g(3-y)}{5}\right)t_1$ (5)

After the string is cut:



$yg - T_3 = ya'$ & $T_3 = 4a'$, so that $yg - 4a' = ya'$

and hence $a' = \frac{yg}{y+4}$

Suppose that the block takes a further time t_2 to come to rest.

Then (from "$v = u + at$") $0 = -\left(\frac{g(3-y)}{5}\right) t_1 + \frac{ygt_2}{y+4}$ (from (5))

so that $t_2 = \frac{(3-y)(y+4)t_1}{5y}$

and hence, from (4), the required time

$= t_1 + t_2 = \sqrt{\frac{10d}{g(3-y)}}\{1 + \frac{(3-y)(y+4)}{5y}\}$

$= \sqrt{\frac{d}{5g}} f(y)$

where $f(y) = \sqrt{\frac{50}{3-y}} + \sqrt{\frac{50}{3-y}} \left[\frac{(3-y)(y+4)}{5y}\right]$

$= \sqrt{\frac{100}{6-2y}} + \sqrt{\frac{100}{6-2y}} \left[\frac{(6-2y)\left(1+\frac{4}{y}\right)}{10}\right]$

$= \frac{10}{\sqrt{6-2y}} + \left(1 + \frac{4}{y}\right)\sqrt{6-2y}$, as required

$f'(y) = 10\left(-\frac{1}{2}\right)(6-2y)^{-\frac{3}{2}}(-2)$

$+4(-1)y^{-2}\sqrt{6-2y} + \left(1+\frac{4}{y}\right)\left(\frac{1}{2}\right)(6-2y)^{-\frac{1}{2}}(-2)$

$= y^{-2}(6-2y)^{-\frac{3}{2}}\{10y^2 - 4(6-2y)^2 - (y^2+4y)(6-2y)\}$

Then $f'(y) = 0 \Rightarrow 10y^2 - 4(6-2y)^2 - (y^2+4y)(6-2y) = 0$

$\Rightarrow 2y^3 + y^2(10-16-6+8) + y(96-24) - 144 = 0$

$\Rightarrow 2y^3 - 4y^2 + 72y - 144 = 0$

$\Rightarrow g(y)[say] = y^3 - 2y^2 + 36y - 72 = 0$

Noting that $g(2) = 0$,

$g(y) = (y-2)(y^2+36)$,

so that $y = 2$ is the only solution.