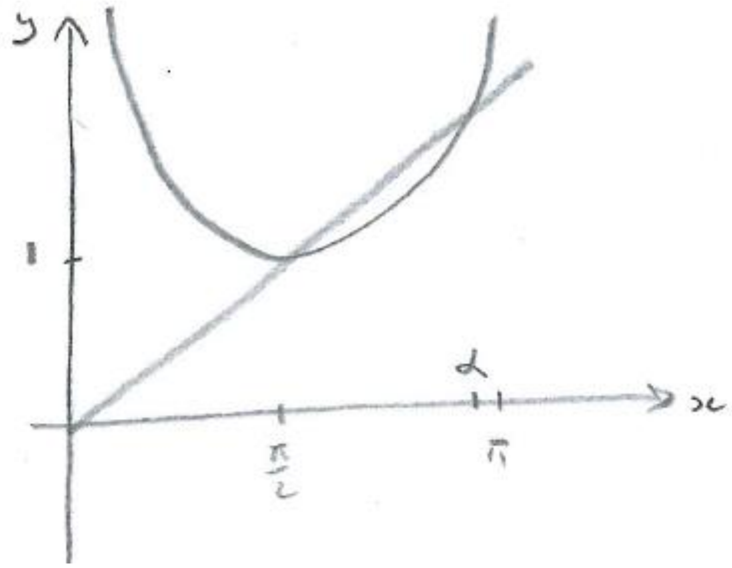
STEP 2006, Paper 1, Q7 - Solution (2 pages; 14/5/18)



(i) From the graph, $cosecx = \frac{2x}{\pi}$ for $x = \frac{\pi}{2}$ and one other value between $\frac{\pi}{2}$ & π ; so $xsinx = \frac{\pi}{2}$ for these two values.

From the graph, for $\frac{\pi}{2} < x < \alpha$, $cosecx < \frac{2x}{\pi}$, so that

$\frac{\pi}{2} < xsinx$ (as $sinx > 0$)

ie $\left|xsinx - \frac{\pi}{2}\right| = xsinx - \frac{\pi}{2}$ for $\frac{\pi}{2} < x < \alpha$

and $\left|xsinx - \frac{\pi}{2}\right| = -(xsinx - \frac{\pi}{2})$ for $\alpha < x < \pi$

So $\int_{\frac{\pi}{2}}^{\pi} \left|xsinx - \frac{\pi}{2}\right| dx = \int_{\frac{\pi}{2}}^{\alpha} xsinx - \frac{\pi}{2}\, dx + \int_{\alpha}^{\pi} \frac{\pi}{2} - xsinx\, dx$

$= [x(-cosx)]_{\frac{\pi}{2}}^{\alpha} - \int_{\frac{\pi}{2}}^{\alpha}(-cosx)dx - \frac{\pi}{2}\left(\alpha - \frac{\pi}{2}\right) + \frac{\pi}{2}(\pi - \alpha)$

$-[x(-cosx)]_{\alpha}^{\pi} + \int_{\alpha}^{\pi}(-cosx)dx$

$= \alpha(-cos\alpha) + [sinx]_{\frac{\pi}{2}}^{\alpha} - \pi\alpha + \left(\frac{\pi}{2}\right)^2 + \frac{\pi^2}{2}$

$+(-\pi - \alpha cos\alpha) - [sinx]_{\alpha}^{\pi}$

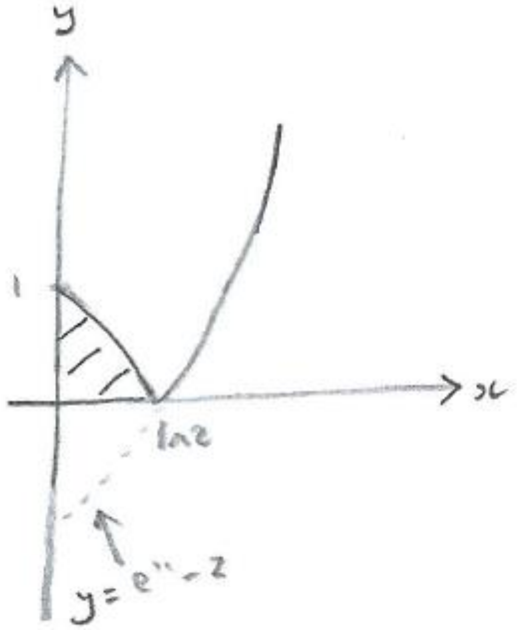
$= -2\alpha cos\alpha + sin\alpha - 1 - \pi\alpha + \frac{3\pi^2}{4} - \pi + sin\alpha$

$= 2sin\alpha + \frac{3\pi^2}{4} - \alpha\pi - \pi - 2\alpha cos\alpha - 1$, as required.

(ii) [As $x > 0$, the $|e^x - 1|$ term seems to be a bit of a red herring; ie it can just be replaced with $e^x - 1$]

As $(e^x - 1) - 1 = 0$ when $x = ln2$, we want $\int_0^{ln2} -(e^x - 2)\, dx$

(see diagram)



$= [2x - e^x]_0^{ln2} = 2ln2 - 2 - (-1) = ln(2^2) - 1 = ln4 - 1$