STEP 2006, Paper 1, Q3 - Solution (5 pages; 27/3/25)

3 In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

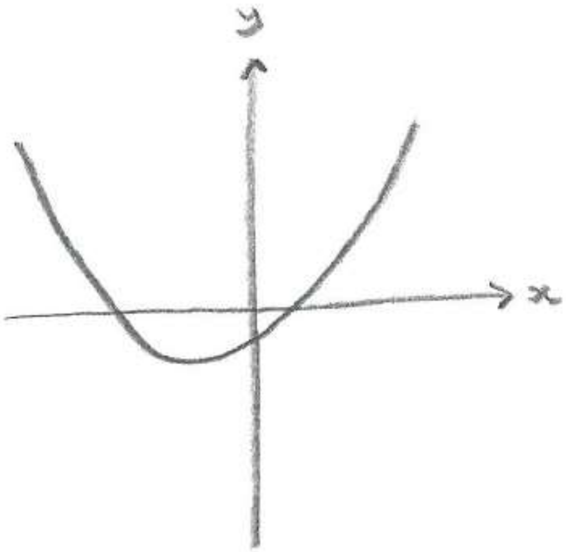
(ii) Determine necessary and sufficient conditions for the equation $x^2 + bx + c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.

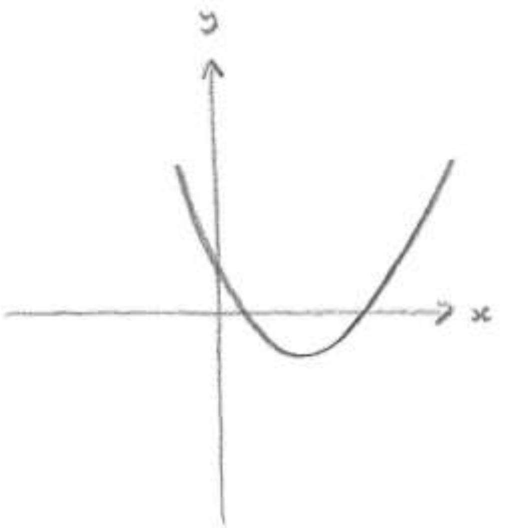
(i) **1st Part**

If $c < 0$, then the graph of $y = x^2 + bx + c$ will either be as in the diagram below, with the minimum at a negative value of x, or similarly at a positive value of x, or at $x = 0$. In each case, we can see that the graph crosses the x −axis at two distinct points.

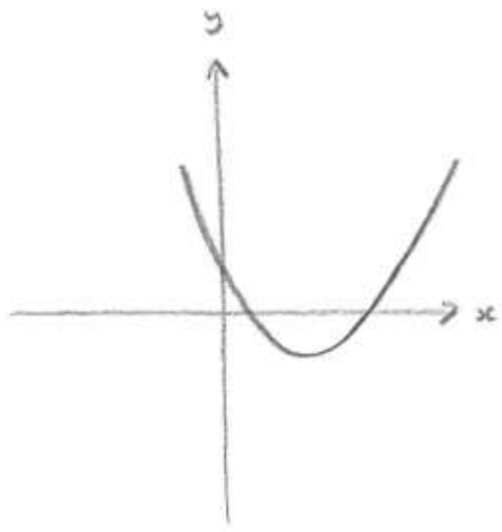


2nd Part

As shown in the diagram below, $c < 0$ is not a necessary condition for distinct real roots.



(ii) The diagram shows the situation where there are distinct positive roots.



A necessary condition is that $c > 0$.

The turning point has x-coordinate $-\frac{b}{2}$

So another necessary condition is that $b < 0$.

In order for there to be two distinct roots, a further necessary condition is that $b^2 - 4c > 0$

These 3 conditions are sufficient because they ensure that the graph appears as in the diagram.

(iii) **1st Part**

Considering the gradient of $y = x^3 + px + q$:

$\frac{dy}{dx} = 3x^2 + p$; so when $p > 0, \frac{dy}{dx} > 0$ for all x; ie y is strictly increasing, and so crosses the x-axis once only.

Thus there is 1 positive real root (and 2 complex roots) of

$x^3 + px + q = 0$ when $p > 0$ and $q < 0$ (as the y -intercept is negative, so that the curve crosses the x-axis when $x > 0$).

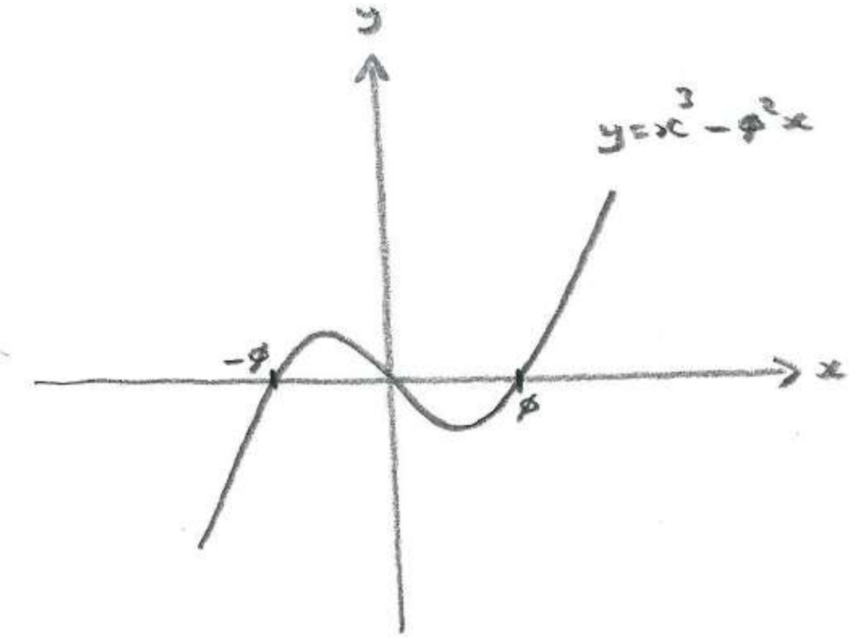
2nd Part

If $p < 0$ and $q < 0$, let $p = -\phi^2$ (where $\phi > 0$) and consider the simpler graph:

$y = x^3 + px = x^3 - \phi^2 x = x(x - \phi)(x + \phi)$

The number of real roots of $x^3 + px + q = 0$ will then depend on the size of q relative to the height of the local maximum of

$y = x^3 - \phi^2 x$ (see diagram below).



The maximum occurs when $\frac{dy}{dx} = 0$; ie when $3x^2 - \phi^2 = 0$,

and $x = -\frac{\phi}{\sqrt{3}}$; when $y = x(x^2 - \phi^2) = -\frac{\phi}{\sqrt{3}}\left(\frac{\phi^2}{3} - \phi^2\right) = \frac{2\phi^3}{3\sqrt{3}}$

So, when $|q| < \frac{2\phi^3}{3\sqrt{3}}$, there will be 3 distinct real roots.

When $|q| = \frac{2\phi^3}{3\sqrt{3}}$, there will be 3 real roots, of which 2 are repeated.

When $|q| > \frac{2\phi^3}{3\sqrt{3}}$, there will be 1 real root (and 2 complex roots).

Now, $4p^3 + 27q^2 > 0 \Leftrightarrow 27q^2 > -4p^3 \Leftrightarrow 3\sqrt{3}|q| > 2\phi^3$

$\Leftrightarrow |q| > \frac{2\phi^3}{3\sqrt{3}}$, and similarly for the other cases.

So, if (a) $4p^3 + 27q^2 > 0$, there will be 1 real root (and 2 complex roots);

if (b) $4p^3 + 27q^2 = 0$, there will be 3 real roots, of which 2 are repeated,

and if (c) $4p^3 + 27q^2 < 0$, there will be 3 distinct real roots.

From the above diagram, we can also comment on the signs of the roots:

For (a), the root is positive.

For (b), the repeated root is negative and the other one is positive.

For (c), 2 of the roots are negative and the other is positive.