STEP 2006, Paper 1, Q12 – Solution (2 pages; 15/5/18)

Let D & E be the roads that have 2 sections that could be blocked, and F the road with one such section.

P(O is cut off from C)

= P(D is blocked)P(E is blocked)P(F is blocked)

P(D is blocked) = P(E is blocked) = 1 −P(road is not blocked)

$= 1 - (1-p)(1-p) = 2p - p^2$

Hence P(O is cut off from C) $= (2p-p^2)^2 p = p^3(2-p)^2$, as required.

P(cut off via other roads | a clear road has been selected)

$$= \frac{P(clear\ road\ selected\ and\ cut\ off\ via\ other\ roads)}{P(clear\ road\ selected)}$$

$P(clear\ road\ selected)$

$= P(D\ or\ E\ selected)P(it\ is\ clear) + P(F\ selected)P(it\ is\ clear)$

$= \frac{2}{3}(1-p)^2 + \frac{1}{3}(1-p) = \frac{1}{3}(1-p)(2-2p+1)$

$= \frac{1}{3}(1-p)(3-2p)$

$P(clear\ road\ selected\ and\ cut\ off\ via\ other\ roads)$

$= P(D\ chosen)P(D\ clear)P(E\ blocked)P(F\ blocked)$

$+P(E\ chosen)P(E\ clear)P(D\ blocked)P(F\ blocked)$

$+P(F\ chosen)P(F\ clear)P(D\ blocked)P(E\ blocked)$

As the 1st two terms are equal,

Prob. $= (2)\left(\frac{1}{3}\right)(1-p)^2[1-(1-p)^2]p$

$+\frac{1}{3}(1-p)[1-(1-p)^2]^2$

$= \frac{1}{3}(1-p)(2p-p^2)\{2(1-p)p+(2p-p^2)\}$

$= \frac{1}{3}p(1-p)(2-p)\{4p-3p^2\}$

$= \frac{1}{3}p^2(1-p)(2-p)(4-3p)$

So required prob. $= \frac{p^2(2-p)(4-3p)}{3-2p}$

As $p \to 1$, prob. $\to \frac{1(2-1)(4-3)}{3-2} = 1$

[Note: There is a (university level) theorem which says that

$\lim\limits_{x\to L}\frac{A(x)}{B(x)} = \frac{\lim\limits_{x\to L}A(x)}{\lim\limits_{x\to L}B(x)}$, provided that $\lim\limits_{x\to L}A(x)$ & $\lim\limits_{x\to L}B(x)$ are both constants]

As $p \to 1$, we would expect that, were we lucky enough to find an unblocked road, it would be increasingly likely that the other two roads would both be blocked.