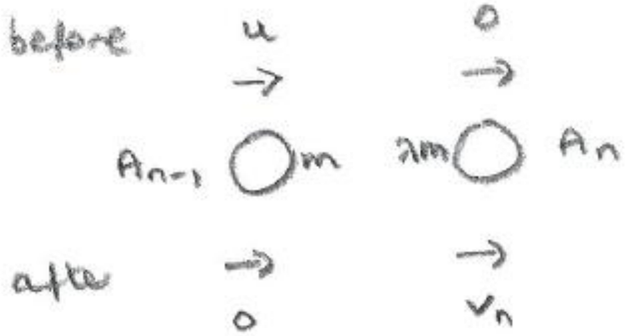
STEP 2006, Paper 1, Q11 – Solution (3 pages; 15/5/18)

All velocities are > 0

(i)



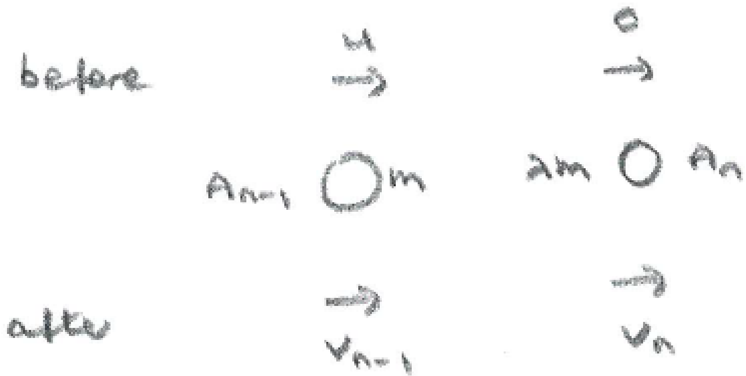
As the total momentum is > 0, if there is just one particle moving at the end, it must be A_n.

Then the momentum and kinetic energy of A_n must equal that of A_0.

So $\lambda m v_n = mu$ & $\frac{1}{2}\lambda m {v_n}^2 = \frac{1}{2}mu^2$

Hence $\lambda {v_n}^2 = (\lambda v_n)^2$ and $\lambda = \lambda^2$, so that $\lambda = 0$ or 1, which contradicts the fact that $\lambda > 1$

(ii)



If only A_{n-1} & A_n are moving at the end, then, by conservation of momentum (CoM), the initial velocity of A_{n-1} is u.

Then by CoM, $mu = mv_{n-1} + \lambda m v_n$, so that $u = v_{n-1} + \lambda v_n$ (1)

And by Conservation of energy (CoE),

$\frac{1}{2}mu^2 = \frac{1}{2}m{v_{n-1}}^2 + \frac{1}{2}\lambda m{v_n}^2$, so that $u^2 = {v_{n-1}}^2 + \lambda {v_n}^2$ (2)

Then (1) & (2) $\Rightarrow {v_{n-1}}^2 + 2\lambda v_{n-1}v_n + \lambda^2 {v_n}^2 = {v_{n-1}}^2 + \lambda {v_n}^2$

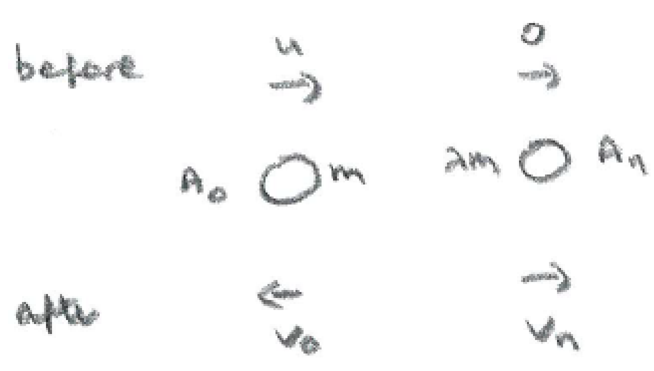
and $2\lambda v_{n-1}v_n + \lambda^2 {v_n}^2 - \lambda {v_n}^2 = 0$,

so that $2v_{n-1} + \lambda v_n - v_n = 0$ (as λ & v_n are $\neq 0$)

Then $v_n = \frac{-2v_{n-1}}{\lambda - 1}$, which contradicts the assumption that v_{n-1} & v_n are both > 0 (since $\lambda > 0$).

(iii) Even if A_{n-2} is moving as well as A_{n-1} & A_n, the same argument can be used as in (ii), replacing u with the initial velocity of A_{n-1}, since momentum and energy are conserved for the collision between A_{n-1} & A_n.

(iv)



By (ii), the 2 particles can't be A_{n-1} & A_n, and so must be A_0 & A_n

By CoM, $mu = m(-v_0) + \lambda m v_n$, so that $u = \lambda v_n - v_0$ (3)

By CoE, $\frac{1}{2}mu^2 = \frac{1}{2}m{v_0}^2 + \frac{1}{2}\lambda m{v_n}^2$, so that $u^2 = {v_0}^2 + \lambda{v_n}^2$ (4)

Then (3) & (4) $\Rightarrow \lambda^2{v_n}^2 - 2\lambda v_n v_0 + {v_0}^2 = {v_0}^2 + \lambda{v_n}^2$,

so that $\lambda^2{v_n}^2 - 2\lambda v_n v_0 - \lambda{v_n}^2 = 0$

and hence $\lambda v_n - 2v_0 - v_n = 0$ (as λ & v_n are both $\neq 0$)

Then from (3), $v_0 = \lambda v_n - u$,

so that $\lambda v_n - 2(\lambda v_n - u) - v_n = 0$

and hence $v_n(-\lambda - 1) + 2u = 0$ and $v_n = \frac{2u}{\lambda+1}$

Then from (3) again, $v_0 = \lambda v_n - u = \lambda\left(\frac{2u}{\lambda+1}\right) - u$

$= \frac{u}{\lambda+1}\big(2\lambda - (\lambda+1)\big) = \frac{u(\lambda-1)}{\lambda+1}$