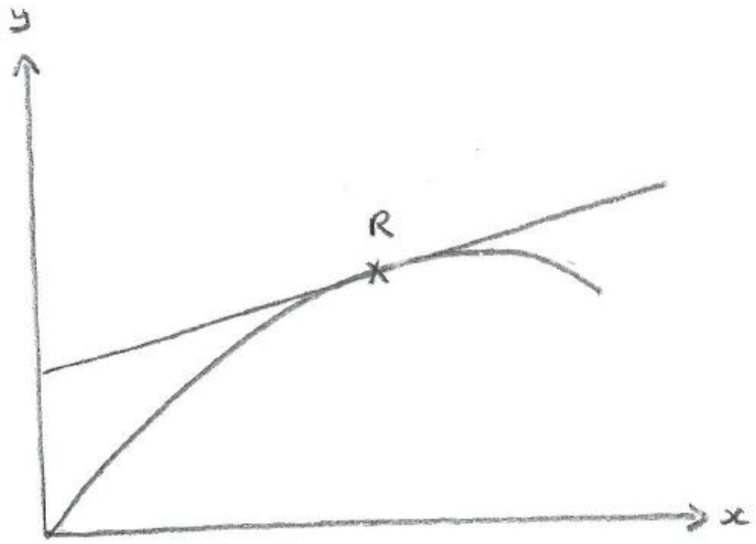
STEP 2006, Paper 1, Q10 – Solution (3 pages; 15/5/18)



$\binom{x}{y} = \begin{pmatrix} \frac{V}{\sqrt{2}} \\ \frac{V}{\sqrt{2}} \end{pmatrix} t + \frac{1}{2}\binom{0}{-g} t^2$,

so that $x = \frac{Vt}{\sqrt{2}}$ & $y = \frac{Vt}{\sqrt{2}} - \frac{gt^2}{2}$

and hence $y = x - \frac{g}{2}\left(\frac{x\sqrt{2}}{V}\right)^2 = x - \frac{gx^2}{V^2}$

At points where the curve intersects the line $y = xtan\alpha + b$,

$x - \frac{gx^2}{V^2} = xtan\alpha + b$

$\Rightarrow gx^2 + xV^2(tan\alpha - 1) + V^2 b \quad (1)$

If the particle just touches the roof, (1) has repeated roots and so $\Delta = 0$:

$V^4(tan\alpha - 1)^2 - 4gV^2 b = 0$

$\Rightarrow V^2|tan\alpha - 1| = 2V\sqrt{gb}$

In order for the particle to reach the roof, $\alpha < 45°$,

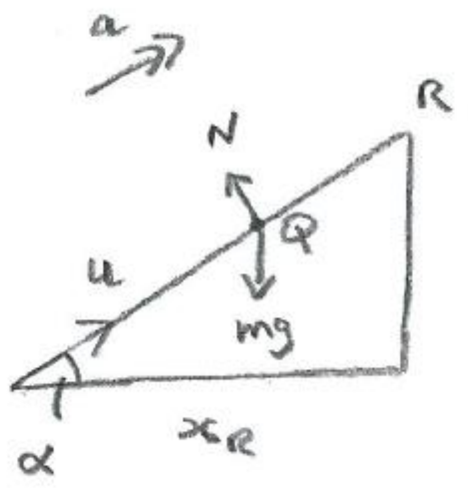
and so $V(1 - tan\alpha) = 2\sqrt{gb}$

giving $V(-1 + tan\alpha) = -2\sqrt{bg}$, as required

If T is the required time, $x_R = \frac{V}{\sqrt{2}}T$, where x_R is the x coordinate of R, which is the repeated root of (1),

so that $x_R = \frac{-V^2(tan\alpha-1)}{2g}$ (2) and $T = \frac{V(1-tan\alpha)}{g\sqrt{2}}$ (3)

In order for P and Q to meet, Q must take time T to reach R.



Referring to the diagram above,

N2L along the slope gives $-mgsin\alpha = ma$,

so that $a = -gsin\alpha$

and $x_R = \left(UT - \frac{1}{2}gsin\alpha.T^2\right)cos\alpha$ (4)

Combining (2), (3) & (4) gives:

$$\frac{-V^2(tan\alpha-1)}{2g}=\frac{V(1-tan\alpha)}{g\sqrt{2}}\left(U-\frac{1}{2}gsin\alpha\left(\frac{V(1-tan\alpha)}{g\sqrt{2}}\right)\right)cos\alpha$$

$$\Rightarrow V=\sqrt{2}\left(U-\frac{V(1-tan\alpha)sin\alpha}{2\sqrt{2}}\right)cos\alpha$$

$$\Rightarrow V+\frac{V(1-tan\alpha)sin\alpha cos\alpha}{2}=\sqrt{2}Ucos\alpha$$

$$\Rightarrow 2\sqrt{2}Ucos\alpha=V(2+(1-tan\alpha)sin\alpha cos\alpha)$$

$=V(2+sin\alpha cos\alpha-sin^2\alpha)$, as required