STEP 2005, Paper 3, Q7 - Solution (2 pages; 8/5/20)

As the Examiner's Report mentions, it isn't essential to use the result proved at the start when answering parts (i) & (ii). In (ii), although a non-standard integral is involved (assuming the initial result is used), the most natural substitution (ie $z^2 = u + 1$) turns out to work. Also for (ii), it is odd that the H&As don't consider separately the case where $n = 2$ (apart from mentioning that n can't equal 2). Normally you would be expected to do this.

Also, strictly speaking, $\sqrt{x^n + x^2} = |x|\sqrt{x^{n-2} + 1}$.

Let $u = x^m$, so that $du = mx^{m-1}dx = \frac{mu}{x}dx \Rightarrow \frac{m}{x}dx = \frac{1}{u}du$

Hence $\int \frac{m}{xf(x^m)}dx = \int \frac{1}{u}.\frac{1}{f(u)}du = F(x^m) + c$

(i) Let $I = \int \frac{1}{x^n - x}\, dx = \int \frac{1}{x(x^{n-1}-1)}dx$

Then let $f(u) = u - 1$ & $m = n - 1$ $(n \neq 1)$

Then $I = \frac{1}{m}\int \frac{m}{xf(x^m)}dx = \frac{1}{n-1}F(x^{n-1}) + c$

where $F(u) = \int \frac{1}{u(u-1)}du = \int -\frac{1}{u} + \frac{1}{u-1}du = \ln|\frac{u-1}{u}|$

Hence $I = \frac{1}{n-1}\ln\left|\frac{x^{n-1}-1}{x^{n-1}}\right| + c$

(ii) Let $J = \int \frac{1}{\sqrt{x^n + x^2}}\, dx = \int \frac{1}{x\sqrt{x^{n-2}+1}}dx$

Then let $f(u) = \sqrt{u + 1}$ & $m = n - 2$, assuming that $n \neq 2$

Then $J = \frac{1}{m}\int \frac{m}{xf(x^m)}\, dx = \frac{1}{n-2}F(x^{n-2}) + c$,

where $F(u) = \int \frac{1}{u\sqrt{u+1}}du$

Let $z^2 = u + 1$, so that $2zdz = du$

and $F(u) = \int \frac{2z}{(z^2-1)z} dz = \int \frac{1}{z-1} - \frac{1}{z+1} dz$

$= \ln\left|\frac{z-1}{z+1}\right| = \ln|\frac{\sqrt{u+1}-1}{\sqrt{u+1}-1}|$

Hence $J = \frac{1}{n-2}\ln\left|\frac{\sqrt{x^{n-2}+1}-1}{\sqrt{x^{n-2}+1}+1}\right| + c$, when $n \neq 2$

And when $n = 2$, $J = \frac{1}{\sqrt{2}}\int \frac{1}{x} dx = \frac{1}{\sqrt{2}}\ln|x| + c$