STEP 2005, Paper 3 - Notes (4 pages; 9/5/20)

See separate documents for Sol'ns.

(N): brief comment only

(Sol'n): sol'n to part(s) of q'n

1	2	3	4	5	6	7	8
N	Sol'n	Sol'n		N	(Sol'n)	Sol'n	Sol'n

9	10	11		12	13	14
N					N	

Q1 Surprise is expressed in the examiner's report that this question was so unpopular, but it goes on to point out that "graph sketches in STEP papers will often require considerable working, such as determining turning points and their nature, even if this is not explicitly indicated in the question". These sketch questions are generally not good value for money; especially trig. graphs requiring consideration of eg sin x, where x is not a simple multiple of π.

Also, "if and only if" proofs require both "if" and "only if" to be considered; though it is often sufficient to use ⇔ throughout (provided that this is clearly true).

Q5

For the 1st part, in the official Hints & Answers, use is being made of the following form of the equation of a line:

$y - mx = y_1 - mx_1$ (where $x_1 = \frac{m-b}{2a}$ etc) (though the previous simplification of y_1 to $\frac{m^2-b^2}{4a}+c$ doesn't seem to have been used)

Alternative method to show that , when $a_1 \neq b_1$, the two curves have exactly one common tangent if and only if they touch each other:

The discriminant of the quadratic eq'n in m has to be zero for there to be

exactly one common tangent; ie we require

$4(a_1b_2-a_2b_1)^2-4(a_2-a_1)\{4a_1a_2(c_2-c_1)+a_2{b_1}^2-a_1{b_2}^2\}=0$ (1)

For the curves to touch each other, there must be exactly one solution to

$a_1x^2+b_1x+c_1=a_2x^2+b_2x+c_2$;

ie the discriminant of this quadratic eq'n must also be zero, so that

$(b_1-b_2)^2-4(a_1-a_2)(c_1-c_2)=0$ (2)

The method is then to first show that the LHS of (1) reduces to zero when use is made of (2) [ie to demonstrate that (2) ⇒ (1)], and then (by rearranging things) to show that if (2) does **not** hold, then (1) also doesn't hold; ie that $(2)' \Rightarrow (1)'$, which is equivalent to (1) ⇒ (2).

Q6 (2nd part)

Writing $b=a^2$ & $c=a^3coshT$,

rtp (result to prove): $2acosh\left(\frac{T}{3}\right)=u+\frac{b}{u}$ (1) , where $u = \left(c + \sqrt{c^2 - b^3}\right)^{\frac{1}{3}}$ (2)

(1) can be rewritten as $a\left(e^{\frac{T}{3}} + e^{-\frac{T}{3}}\right) = u + \frac{a^2}{u}$ or $e^{\frac{T}{3}} + e^{-\frac{T}{3}} = \frac{u}{a} + \frac{a}{u}$

So (1) will be satisfied if $\frac{u}{a} = e^{\frac{T}{3}}$; ie if $u^3 = a^3 e^T$ (3)

The form of (2) suggests that u^3 might be the sol'n of a quadratic eq'n.

$c = a^3 coshT$ can be expressed as a quadratic e'qn in e^T:

$\frac{2c}{a^3} = e^T + e^{-T}$

$\Rightarrow e^{2T} - \frac{2ce^T}{a^3} + 1 = 0$

$\Rightarrow e^T = \frac{c}{a^3} \pm \frac{1}{2}\sqrt{\frac{4c^2}{a^6} - 4}$

$\Rightarrow a^3 e^T = c \pm \sqrt{c^2 - a^6}$ (4)

The question asks us to find just one of the roots, so taking the +ve root in (4), $a^3 e^T = u^3$, which is (3).

Q9

This topic (collisions) can only really involve conservation of momentum and Newton's law of impact, as well as the relationship between distance, velocity and time. A shortcut can be expected when generalising to later collisions (ie noting that the situation is repeated, but with a scale factor applied). As usual with many Mechanics questions, there is a fair amount of algebra (though nothing demanding).

A relatively painless way of eliminating T from the following equations (where T is the time between the 1st & 2nd collisions, and x is the distance from the wall for the 2nd collision)

$$d - ev(1-e)T = x \quad \& \; 2e^2v\left(T - \frac{d}{2ev}\right) = x$$

is simply to make T the subject of each equation and then equate the two expressions.

Q13

For part (i), the official sol'ns seem to suggest that the result $\sum_{r=0}^{\infty} rp^{r-1} = \frac{1}{(p-1)^2}$ can just be quoted. The result can be derived from the mean of a Geometric variable, and it might be best to show that (if it isn't being obtained by differentiating $\sum_{r=0}^{\infty} p^r$).

In part (ii), there is the common dilemma for STEP of deciding to what extent the previous method can be applied, or whether a different approach is needed.

It turns out, perhaps surprisingly, that the same method can be used; and in any case the alternative of conditioning on the point at which the 0 card is drawn would appear to produce a complicated double summation involving factorials, which doesn't seem to be capable of being evaluated.

The moral (with hindsight) is not to waste time considering anything complicated. The extra difficulty for part (ii) is not really in modifying the method, but applying the method where you aren't shown how to start the problem.