STEP 2005, Paper 2, Q6 - Solution (3 pages; 11/5/18)

(i) Binomial expansions give:

$(1-x)^{-1} = 1 + x + x^2 + \cdots$

[$(1-x)^{-1}$ is the sum to infinity of the GP with 1st term 1 and common ratio x]

General term: x^r

$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \dots$

General term: $(r+1)x^r$

$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3$

$+ \frac{(-3)(-4)(-5)(-6)}{4!}(-x)^4 + \cdots$

General term: $\frac{(r+1)(r+2)}{2}x^r$

Letting $x = \frac{1}{2}$ (so that $|x| < 1$),

$\sum_{n=1}^{\infty} n2^{-n} = \sum_{r=1}^{\infty} rx^r = x\sum_{r=1}^{\infty} rx^{r-1} = x\sum_{R=0}^{\infty}(R+1)x^R$

where $R = r - 1$

$= x(1-x)^{-2}$

$= \frac{1}{2}(1-\frac{1}{2})^{-2} = \frac{1}{2}(4) = 2$

$\sum_{n=1}^{\infty} n^2 2^{-n} = \sum_{n=1}^{\infty} n(n-1)2^{-n} + \sum_{n=1}^{\infty} n2^{-n}$

$= \{\sum_{r=0}^{\infty}(r+2)(r+1)x^{r+2}\} + 2$

with $x = \frac{1}{2}$ & $r + 2 = n$ (since $\sum_{n=1}^{\infty} n(n-1)2^{-n} = \sum_{n=2}^{\infty} n(n-1)2^{-n}$)

$= \{x^2 \sum_{r=0}^{\infty} (r+2)(r+1)x^r\} + 2$

$= \left(\frac{1}{2}\right)^2 (2) \left(1 - \frac{1}{2}\right)^{-3} + 2$

$= \frac{1}{2}(8) + 2 = 6$

(ii) The Binomial expansion of $(1-x)^{-\frac{1}{2}}$ is

$1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + \cdots$ for $|x| < 1$

The general term is $\frac{(1)(3)(5)\ldots(2r-1)}{2^r r!} x^r = \frac{(2r)!}{(2)(4)(6)\ldots(2r)2^r r!} x^r$

$= \frac{(2r)!}{r!2^r.2^r r!} x^r = \frac{(2r)!}{(r!)^2 2^{2r}} x^r$

so that, replacing r with n, $(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}} x^n$ for $|x| < 1$, as required.

$x = 1/3$ then gives $(1 - 1/3)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}} 3^{-n}$

or $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n} 3^n} = \sqrt{\frac{3}{2}}$

To obtain the n in $\sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^n}$, we can first of all differentiate

$(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}} x^n$ to give

$(-\frac{1}{2})(1-x)^{-\frac{3}{2}}(-1) = \sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n}} x^{n-1}$

Then setting $x = 1/3$ gives $(1/2)(1-\frac{1}{3})^{-3/2} = \sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^{n-1}}$

Finally, $\sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^{n}} = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)^{-\frac{3}{2}} = \frac{1}{6}\sqrt{\frac{27}{8}} = \frac{1}{6}\left(\frac{3}{2}\right)\sqrt{\frac{3}{2}} = \frac{1}{4}\sqrt{\frac{3}{2}}$