STEP 2005, Paper 1, Q5 – Solution (2 pages; 9/5/18)

(i) For $k \neq 0$, $\int_0^1 (x+1)^{k-1} dx = \left[\frac{1}{k}(x+1)^k\right]_0^1 = \frac{1}{k}(2^k - 1)$

For $k = 0$, $\int_0^1 \frac{1}{x+1} dx = [\ln(x+1)]_0^1 = ln2$

As $(x+1)^{k-1}$ is a continuous function of k around $k = 0$, for a given value of x [ie it doesn't jump at $k = 0$], it follows that the two integrals will get closer as k approaches 0, since they represent areas under curves of functions that become progressively closer.

So $\frac{1}{k}(2^k - 1) \approx ln2$ when $k \approx 0$.

[It isn't really possible at A Level to give a much more rigorous argument. In this sort of situation, the examiners are likely to be just testing a candidate's awareness of the issue. Having said all this, the official solutions don't seem to address the issue at all - which is unusual, as STEP solutions regularly discuss such refinements. But you can never be entirely sure what the examiners are going to home in on.]

[Although it may seem a trivial point, it is necessary to explicitly state the conclusion, as otherwise the examiner may not be sure that you know that the solution is complete! (Perhaps more applicable in more complicated situations.)]

(ii) [The official solutions imply that candidates are likely to notice that

$x(x+1)^m$ can usefully be rewritten as $(x+1-1)(x+1)^m$ (though it's a useful trick to recall).]

Parts is possible. In general though, Parts has a couple of drawbacks:

(i) You can't always be sure that it will lead anywhere. In the case of definite integrals, it is possible to end up proving that $I = I$ (where I is the original integral).

(ii) It can require more than one application of the Parts formula.

So it is usually a good idea to consider the other possible approaches first. Apart from the possibility of the integral being in the formulae booklet, these are:

(a) looking for a rearrangement of the integrand (as suggested in the official solutions for this question); this includes breaking down into partial fractions

(b) a substitution

(i) $u = f(x)$, where the integrand can be written as $f'(x)g(f(x))$, provided that $g(u)$ can be integrated

(eg $u = sinx$, where the integrand is $cosxsin^7x$)

(ii) eg $u = x + 1$, to simplify $x(x+1)^m$ (ie as in this question)

(iii) a specialist substitution such as $t = tan\left(\frac{\theta}{2}\right)$, which can reduce an integrand involving trig. functions to one involving powers of t

[See "Integration Methods" for further details.]

So in this case we can write $u = x + 1$, to give

$\int_0^1 x(x+1)^m dx = \int_1^2 (u-1)u^m du = \int_1^2 u^{m+1} - u^m du$

The 3 cases to consider are $m = -1, m = -2$ and other m.

For $m = -1$, $\int_1^2 u^{m+1} - u^m du = \int_1^2 1 - \frac{1}{u} du = [u - lnu]_1^2$

$= (2 - ln2) - (1 - 0) = 1 - ln2$

For $m = -2$, $\int_1^2 u^{m+1} - u^m du = \int_1^2 \frac{1}{u} - u^{-2} du = [lnu + u^{-1}]_1^2$

$= \left(ln2 + \frac{1}{2}\right) - (0 + 1) = ln2 - \frac{1}{2}$

For other m, $\int_1^2 u^{m+1} - u^m du = \left[\frac{1}{m+2}u^{m+2} - \frac{1}{m+1}u^{m+1}\right]_1^2$

$= \left(\frac{2^{m+2}}{m+2} - \frac{2^{m+1}}{m+1}\right) - \left(\frac{1}{m+2} - \frac{1}{m+1}\right)$

$= \frac{2^{m+1}}{(m+1)(m+2)}[2(m+1) - (m+2)]$

$- \frac{1}{(m+1)(m+2)}[(m+1) - (m+2)]$

$= \frac{2^{m+1}(m)+1}{(m+1)(m+2)}$